# Evaluation of Artin's Constant and the Twin-Prime Constant

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**1. Introduction.** In 1914 A. J. C. Cunningham [1] investigated the number of primes having the integers between 2 and 12 (exclusive of powers) as primitive roots. In particular, for the integers 2 and 10 he tabulated these counts for each of the first ten myriads, and deduced empirically that the density of such primes in the set of all primes in a given interval lies, effectively, between 0.355 and 0.400.

According to H. Bilharz [2], E. Artin in an oral communication to H. Hasse on 13 September 1927 proposed the question, whether, corresponding to a given nonzero integer $a$, the set of all primes of which $a$ is a primitive root possesses a density. Artin's conjecture as stated by Hasse [3], namely, that corresponding to every integer not a square and different from $-1$, there exists an infinitude of primes having that integer as a primitive root, has never been proved. Furthermore, Artin conjectured that the density of such an infinite set, if it exists, is the same for all non-power integers, and is given by the convergent infinite product

$$A = \prod_{p \geqq 2} \left\{ 1 - \frac{1}{p(p-1)} \right\},$$

taken over all primes $p$. This we have termed Artin's constant.

The twin-prime constant is the first member, $C_2$, of a set of constants defined in 1923 by Hardy and Littlewood [4] by means of the relation

$$C_n = \prod_{p>n} \left\{ \left( \frac{1}{p-1} \right)^{n-1} \left( \frac{p-n}{p-1} \right) \right\}.$$

In that paper (p. 44) Hardy and Littlewood conjectured that

$$P_2(n) \sim 2C_2 \int_2^n \frac{dx}{(\log x)^2},$$

where $P_2(n)$ is the number of prime-pairs less than $n$, and they gave values of the two sides of this asymptotic relation when $n = 10^5(10^5)10^6$.

Apparently unaware of this paper, Sutton [5] in 1937 studied the average distribution of twin primes, using a probabilistic approach, and presented detailed empirical evidence (based on counts to $8 \cdot 10^5$) for the validity of the foregoing asymptotic formula. Subsequently, C. R. Sexton [6] performed an independent count of prime-pairs less than $10^5$, and revealed discrepancies in Sutton's data as well as in the counts made by several other workers in this field.

The most extensive and reliable empirical knowledge of this kind that is available at the present time appears in an unpublished table of D. H. Lehmer [7]. Included in this table are the cumulative totals of prime-pairs in successive millions

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as far as $37 \cdot 10^6$, as well as the corresponding values of $2C_2 li_2(x)$, where $li_2(x)$ represents $\int_2^x (\ln t)^{-2}\, dt$. For example, Lehmer gives 183,728 as the count of prime-pairs less than $37 \cdot 10^6$, and gives 183,582 as the corresponding value of $2C_2 li_2(x)$.

**2. Evaluation of Artin's Constant.** The present calculation of $A$ has been based on the equivalent representation:

$$\ln A = \sum_{p \geqq 2} \ln\left(1 - \frac{\alpha}{p}\right) + \sum_{p \geqq 2} \ln\left(1 - \frac{\beta}{p}\right) - \sum_{p \geqq 2} \ln\left(1 - \frac{1}{p}\right),$$

where $\alpha$ and $\beta$ represent the zeros of the polynomial $p^2 - p - 1$. Hence, if the logarithms are expanded in Maclaurin series and if Newton's formulas are used to express the sums of the same powers of $\alpha$ and $\beta$ in terms of the coefficients of the polynomial $p^2 - p - 1$, there results the series

$$-\ln A = \tfrac{1}{2} a_1 \sum_{p \geqq 2} p^{-2} + \tfrac{1}{3} a_2 \sum_{p \geqq 2} p^{-3} + \tfrac{1}{4} a_3 \sum_{p \geqq 2} p^{-4} + \cdots,$$

where $a_k = a_{k-1} + a_{k-2} + 1$, for $k \geqq 2$, and $a_0 = 0$, $a_1 = 2$.

The calculation was greatly expedited by computing separately the product of the first eleven factors of the product defining $A$, and then applying the preceding transformation to the infinite product consisting of the remaining factors.

Carefully prepared tables of $\sum_{p \geqq 2} p^{-k}$ to 50D for $k = 2(1)167$ have been published by R. Liénard [8]. Subtraction from these data of independently computed and checked sums of reciprocal powers of the first eleven primes to 55D yielded values of $\sum_{p \geqq 37} p^{-k}$ to 50D for $k = 2(1)28$. Direct summation was used to obtain these sums corresponding to $k = 29(1)35$, to at least 54D. Finally, multiplication by the appropriate coefficients $a_k/(k + 1)$ yielded approximations to the individual terms of the modified series for $\ln A$ that were correct to at least 46D.

Accordingly, the following rounded approximation to Artin's constant is believed to be correct to 45D:

$$A = 0.37395\ 58136\ 19202\ 28805\ 47280\ 54346\ 41641\ 51116\ 29249.$$

**3. Evaluation of $C_2$.** The present calculation of the twin-prime constant, $C_2$, was performed in a manner similar to that of Artin's constant.

Since the product representation of $C_2$ can be written in the form

$$C_2 = \prod_{p>2} \left\{1 - \frac{1}{(p-1)^2}\right\},$$

we have

$$-\ln C_2 = \tfrac{1}{2} b_1 \sum_{p>2} p^{-2} + \tfrac{1}{3} b_2 \sum_{p>2} p^{-3} + \tfrac{1}{4} b_3 \sum_{p>2} p^{-4} + \cdots,$$

where $b_k = 2^{k+1} - 2$.

In this calculation the product of the first ten factors was first found directly, and then the previously computed values of $\sum_{p \geqq 37} p^{-k}$ were combined with the appropriate coefficients $b_i$ to yield the terms of the modified series.

The resulting value of $C_2$ truncated at 42D is:

$$C_2 = 0.66016\ 18158\ 46869\ 57392\ 78121\ 10014\ 55577\ 84326\ 23 \cdots .$$

This approximation confirms the accuracy of the 10D values given by Lehmer [7] and by J. C. P. Miller [9]. Sutton's value of 1.3202 for $C_0$ ($= 2C_2$) is too low by about a unit in the last place. Recently Fröberg [10] has determined $C_2$ to 10D, of which the first 8 decimals are correct.

As a by-product of this calculation we deduce a new approximation to $D_\infty$, defined by Rosser [11] as $\lim_{n\to\infty} D_n$ where $D_n$ is determined by the relation

$$\prod_{m=2}^{n} \left(1 - \frac{2}{p_m}\right) = \frac{D_n}{(\log n)^2},$$

where $p_m$ represents the $m$th prime.

The value of $D_\infty$ can be deduced from that of $C_2$ by virtue of the relation [9]

$$D_\infty = 4e^{-2\gamma} C_2,$$

where $\gamma$ is Euler's constant.

Several years ago the writer computed unpublished values of both $e^\gamma$ and $e^{-\gamma}$ to 170S. Consequently, the approximation of $e^{-2\gamma}$ to about 50D was easily accomplished, and the resulting value of $D_\infty$ to 40D was found to be:

$$D_\infty = 0.83242\ 90656\ 61945\ 27803\ 08059\ 43531\ 46557\ 50462 \cdots .$$

This confirms the accuracy of the 12D approximation found by Rosser [11].

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