The recent work of Siegel, Sparrow and Hallman [6] has just come to the attention of the author. These workers have considered this problem in the heat transfer context. They report values of the eigenfunctions and eigenvalues which are in excellent agreement with the more extensive data of the present work.

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Efficient Continued Fraction Approximations To Elementary Functions

By Kurt Spielberg

1. Introduction. This paper describes an application and extension of the work of H. J. Maehly [1] on the rational approximation of arc tan x, and of E. G. Kogbetliantz [2], who developed Maehly's procedure so as to be applicable to the computer programming of elementary transcendental functions.

It is to be shown here that certain modifications, such as the introduction of terms which are easily computed on specific computers, lead to considerable improvements. In particular, the application of the modified method to several elementary functions will be described and corresponding final results will be given. Some of these approximations have been used with great success to develop subroutines for the IBM 704 and 709 computers. Our experience indicates that the method of Maehly and Kogbetliantz, as modified below, is superior to other current numerical procedures.

2. The Modified Method of Maehly and Kogbetliantz. The basic idea made use of by H. J. Maehly in connection with $f(x) = \text{arc tan } x$ is to approximate the function $f(x)$ by a ratio of two Chebyshev sums of order k

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$$f(x) = \sum_{r=0}^{k} a_r \cdot T_r(x) \Big/ \left[1 + \sum_{s=1}^{k} b_s \cdot T_s(x)\right] + H(x) \Big/ \left[1 + \sum_{s=1}^{k} b_s \cdot T_s(x)\right] \equiv f^*(x) + A \tag{1}$$

where $H(x) = \sum_{j=0}^{\infty} A_j^{(k)} \cdot T_{2k+1+j}$, and A is the absolute error.

If the coefficients b_s are small, which will normally be the case for $-1 \leqq x \leqq 1$ and reasonably rapid convergence of the power series for $f(x)$, then the denominator of the error term is close to one over the interval of approximation and $H(x)$ represents the absolute error A with sufficient accuracy (compare [3]). The order k is chosen so as to keep A below the desired upper limit of accuracy. In order to evaluate the unknown coefficients a_r and b_s, one must know the coefficients c_n of the Chebyshev expansion of $f(x)$. A comparison of that expansion with (1) leads, after use of the identity

$$2T_m(x) \cdot T_n(x) \equiv T_{m+n}(x) + T_{m-n}(x) \tag{2}$$

to a set of $2k + 1$ simultaneous linear equations in the $2k + 1$ coefficients a_r and b_s. Additional equations can be established for the coefficients $A_j^{(k)}$ in the error function $H(x)$. For even and odd functions the subscripts of (1) are changed as follows: even, $r \rightarrow 2r$, $s \rightarrow 2s$, $j \rightarrow 2j$, $k \rightarrow 2k + \frac{1}{2}$; odd, $r \rightarrow 2r + 1$, $s \rightarrow 2s$, $j \rightarrow 2j$, $k \rightarrow 2k + 1$.

When this scheme was applied in practice, several additional ideas suggested themselves. They can be listed briefly as follows:

a) Application of the method to functions that can be expressed as ratios of Chebyshev series, such as $\tan(\frac{1}{4}\pi x)$.

b) Use of different degree numerator and denominator polynomials in (1).

c) Consideration of unequal intervals for two complementary expansions, such as $\sin \alpha x$ and $\cos \beta x$, $\alpha + \beta = \pi/2$.

d) Reduction of the relative error by means of a linear correction term in a neighborhood of $x = 0$.

e) Reduction of the error term through introduction of a new parameter that does not lead to a full additional multiplication.

The first three points should become clear in the sequel and need little amplification. Point d is usually of concern for odd functions $f(x)$, if it is desired to obtain accurate results for $g(x) = f(x)/x$ as $x \rightarrow 0$. Chebyshev methods applied to the function $f(x)$ produce an approximation $f^*(x)$ such that the absolute error $|f(x) - f^*(x)|$ is (approximately) minimized over an interval such as $-1 \leqq x \leqq 1$. The relative error of such an approximation, $|f^* - f|/f$, usually becomes intolerably large as x and f approach zero. The natural way to cope with this difficulty would be to apply the Chebyshev approximation method to $g(x)$ rather than $f(x)$. Then the relative error $|f^* - f|/f = |x \cdot g^* - x \cdot g|/x \cdot g = |g^* - g|/g$ is nearly minimized over the interval if $g(x)$ does not vary too much. This approach, however, has the drawback that the improvement in the neighborhood of zero is paid for with a decrease in accuracy in the remainder of the interval. We have found that computer subroutines can easily be written so as to use $f^*(x)$ in most of the interval and

$f^*(x) + x \cdot C$ in an appropriately chosen neighborhood of zero. The choice of the correction term C will be discussed further below.

The most important modification of the method, point e, arises if the special machine characteristics of digital computers are taken into account. The reduction of the error clearly depends on the introduction of the parameters a_i and b_i. Each new parameter allows reduction of one more error term A_i to zero, but also results in an increase of the number of multiplications (or divisions), M, by one. We can, however, achieve a compromise by *restricting* the last parameter to a set of values which may allow the correspondingly introduced multiplication to be performed in a manner particularly suited to the calculator in question. In the case of the IBM 704 the multiplication might be reduced to a shift (a "cheap" multiplication with a power of two), in the case of the IBM 709 to a variable length multiplication requiring little time. Among the *permissible* values for the newly introduced parameter, that one is chosen which allows a maximum reduction of the dominant error term. In other words, one of the residuals in the system of linear equations for the a_i and b_i is not reduced to zero but only below a certain value determined by the desired accuracy.

As an example, we may consider as our newly introduced parameter the coefficient a_5 in

$$(3) \quad f(x) = (a_1 T_1 + a_3 T_3 + a_5 T_5)(1 + b_2 T_2)^{-1} = x \cdot \left(K_1 + K_2 \cdot x^2 + \frac{K_3}{x^2 + K_4}\right).$$

Evidently $K_2 = 8a_5/b_2$. In accordance with the above discussion we restrict K_2 to the form 2^n ($n \cdots$ any integer), so that a_5 becomes restricted to the set of values $b_2 \cdot 2^n$. The integer n is chosen so as to reduce the absolute error as far as possible. Numerical details will be given in Section 3.

The modified procedure of Maehly and Kogbetliantz can now be outlined formally as follows. Given a function $f(x)$, find an approximation $f^*(x)$

$$(4) \qquad f(x) = \frac{\sum_{i=0}^{\infty} c_i T_i(x)}{\sum_{i=0}^{\infty} d_i T_i(x)}, \qquad f^*(x) = \frac{\sum_{i=0}^{l} a_i T_i(x)}{1 + \sum_{i=1}^{m} b_i T_i(x)}.$$

To determine the coefficients a_i and b_i, we consider the absolute error

$$(5) \quad \begin{aligned} f(x) - f^*(x) &= \left[\sum_{i=0}^{\infty} d_i T_i \left(1 + \sum_{i=1}^{m} b_i T_i\right)\right]^{-1} \\ &\quad \cdot \left[\left(1 + \sum_{i=1}^{m} b_i T_i\right) \sum_{i=0}^{\infty} c_i T_i - \sum_{i=0}^{l} a_i T_i \cdot \sum_{i=0}^{\infty} d_i T_i\right] \\ &= \left[\sum_{i=0}^{\infty} d_i T_i \left(1 + \sum_{i=1}^{m} b_i T_i\right)\right]^{-1} \cdot \sum_{i=0}^{\infty} R_i \cdot T_i(x) \end{aligned}$$

The coefficients R_i can be viewed as the residuals, usually rapidly diminishing in magnitude as i increases, of an infinite number of linear equations in the s unknowns x_i: $b_1, b_2 \cdots b_m, a_0, a_1 \cdots a_l$.

$$R_i = \sum_{j=1}^{s} e_{ij}\cdot x_j + f_i, \quad i = 0, 1, 2, 3, \cdots, s-1, \cdots. \tag{6}$$

The e_{ij} and f_i are simple sums formed with the coefficients of the given function $f(x)$, c_i and d_i. For instance, the important special case of a polynomial $f(x)$ gives rise to the following residuals:

$$R_0 = c_0 b_0 + \tfrac{1}{2} \sum_{n=1}^{m} b_n c_n - a_0, \qquad b_0 = 1$$

$$(7) \quad 1 \leqq i \leqq m: \quad R_i = c_i + \tfrac{1}{2} c_0 b_i + \tfrac{1}{2} \sum_{n=1}^{m} b_n \cdot (c_{n+i} + c_{|n-i|}) - a_i$$

$$m < i: \quad R_i = \tfrac{1}{2} \sum_{n=0}^{m} b_n \cdot (c_{n+i} + c_{i-n}) - a_i$$

For $i > l$, the term $-a_i$ is omitted above.

Usually one sets the first s residuals equal to zero so that the final error is determined by the absolute sum of the remaining residuals. Instead of this we endeavor to reduce the first $s + 1$ residuals below a desired bound δ, by introducing the additional parameter a_{l+1}

$$(8) \quad \begin{aligned} R_i &= w_i \cdot \delta, & i &= 0, 1, \cdots, s-1, s \\ x_{s+1} &= a_{l+1} = k \cdot x_\nu, & 1 &\leqq \nu \leqq s. \end{aligned}$$

The w_i are weightfactors which can be chosen arbitrarily, usually as 0 for $i < s$ and as 1 for $i = s$. The choice of x_ν and k depends on the transformation from the rational approximation to the continued fraction. In the example given above x_ν is equal to b_2 and k is chosen to be of the form 2^n.

The residuals R_i clearly become linear combinations of $s + 1$ variables x_i. In view of (8), however, they can be expressed in terms of the first s variables and k.

$$(9) \quad R_i = w_i \cdot \delta = \sum_{j=1}^{s} e_{ij} \cdot x_j + e_{i,s+1} \cdot k \cdot x_\nu + f_i, \quad i = 0, 1, 2, \cdots, s-1, \quad 1 \leqq \nu \leqq s.$$

These equations can now be solved for the x_j in terms of k. As a consequence, one can determine the residual R_s as a function of k

$$(10) \qquad x_i = x_i(k), \quad i = 1, 2, \cdots, s, \qquad R_s = w_s \cdot \delta = f(k).$$

Finally one chooses among the manifold of permissible k, namely of those k which permit the replacement of a multiplication by a more favorable operation, that value which minimizes R_s. It is usually possible to reduce R_s so substantially that the leading term of the absolute error becomes R_{s+1}. Except for a bounded factor stemming from the denominator in (5), the final absolute error is given by

$$(11) \qquad A \cong \delta \cdot \sum_{i=0}^{s} | w_i \cdot T_i | + \sum_{i=s+1}^{\infty} | R_i \cdot T_i | \approx \sum_{i=s+1}^{\infty} | R_i |.$$

It is perhaps of interest to point out that, when applied numerically, this procedure usually produced values of k which did not only reduce R_s but also decreased R_{s+1} in magnitude.

We finally turn our attention again to the correction term discussed in point *d*. Inspection of (11) indicates that in a sufficiently small neighborhood of $x = 0$ one can approximate A by the leading term $R_{s+1} \cdot T_{s+1} \cong R_{s+1} \cdot \text{const} \cdot x$. The size of the interval about zero will primarily depend on the relative magnitude of the two lowest degree terms in the replaced Chebyshev polynomial. For instance, $| R_9 \cdot T_9 | = | R_9 \cdot (256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x) |$ can be replaced by $| 9 \cdot R_9 \cdot x |$ for $|x| \ll 3/\sqrt{120}$, or for a neighborhood of zero in which $120 \cdot R_9 \cdot x^2 < w_9 \cdot \delta$. (Of course, $R_{11} \cdot T_{11}$ must also be below the tolerable error limit.) From a practical standpoint, it is simplest to compute the coefficient of the linear error term as $\lim_{x \to 0} [f(x)/x - f^*(x)/x]$.

3. Applications and Results. The method outlined above was applied to produce rational approximations for the development of optimal elementary function subroutines for the IBM 704 and 709 computers. The resulting routines were tested carefully and have been found to give, within the limits of roundoff error, results of the predicted accuracy.

a) Sine Approximation, 8-Digit Accuracy.

$$\sin^* \alpha x = 2(a_1 T_1 + a_3 T_3 + a_5 T_5)(1 + b_2 T_2)^{-1} \cong z \left(K_1 + \tfrac{1}{4} z^2 + \frac{K_3}{z^2 + K_4} \right) \tag{12}$$

$$\alpha = .3, \qquad z = .3x, \qquad -.3 \leqq z \leqq .3$$

In this case (10) becomes

$$R_3 \cong 10^{-9} \cdot [-.28029 + 10^{-3} \cdot .38076(\pm 2^{n-2} \cdot \alpha^3 + 10^{-3} \cdot .55934)^{-1}].$$

The minimum of R_3 is reached for $n = -3$. The correspondingly attained improvement becomes apparent if one compares $R_3)_{-3}$ with $R_3)_{-\infty}$, the error without correction term.

$$R_3)_{n=-3} \cong .5 \times 10^{-10}, \qquad R_3)_{n=-\infty} \cong .4 \times 10^{-9}$$

In the sequel we shall give the results of our computations as they were calculated, that is to more places than is usually warranted by the accuracy.

$$a_1 = .14838\,52081\,2231 \qquad K_1 = -10^2 \times .19845\,92426\,192$$

$$a_3 = -10^{-3}\,.49269\,95891\,193 \qquad K_3 = 10^4 \times .10429\,26708\,144$$

$$a_5 = 10^{-6}\,.37911\,69631\,734 = 2^{n-3} \cdot \alpha^3 b_2, \qquad K_4 = 10^2 \times .50030\,24548\,541$$

$$b_2 = 10^{-3}\,.89864\,76164\,110$$

$\alpha = .3, \quad n = -3, \quad A$ (absolute error) $\cong 10^{-10} \times .55$, R (relative error) $\cong 14\,A/.2955 \cong 10^{-8} \times .26$

Check: 1) $\lim\,(\sin^* z)/z = .99999\,99989\,9 \quad \text{as} \quad z \to 0$

2) $\sin^*(.15) = .14943\,81324\,75 \cdots, \ \sin(.15) = .14943\,813 \cdots$

b) Cosine Approximation, 8-Digit Accuracy.

$$\cos^* 1.3x = 2(a_0T_0 + a_2T_2 + a_4T_4 + a_6T_6)(1 + b_2T_2 + b_4T_4)^{-1}$$
$$= K_1 - 2z^2 + K_3[z^2 + K_4 + K_5(z^2 + K_6)^{-1}]^{-1}$$

$z = 1.3\,x, \qquad -1.3 \leqq z \leqq 1.3$

(13)

$a_0 =$	$.30812\ 59625\ 215$	$K_1 =$	$10^3 \times .33947\ 71494\ 5237$
$a_2 =$	$-.17646\ 68549\ 891$	$K_3 =$	$-10^5 \times .29702\ 03659\ 0243$
$a_4 =$	$10^{-2} \times .49379\ 28852\ 647$	$K_4 =$	$10^2 \times .57936\ 13928\ 3225$
$a_6 =$	$-10^{-4} \times .34496\ 21183\ 611$	$K_5 =$	$10^4 \times .14546\ 86265\ 7824$
$b_2 =$	$10^{-1} \times .20951\ 16328\ 571$	$K_6 =$	$10^2 \times .48789\ 05740\ 6695$
$b_4 =$	$10^{-4} \times .81647\ 83866\ 535$		

In this example equations (8) and (10) take the form:

$$a_6 = 2^{n-2}\cdot(1.3)^2\cdot b_4\,, \qquad R_5 \cong 10^{-8}(-.1490 + 2^{n-1}.3608)(.2862 + 2^{n-1}1.667)^{-1}$$
$$R_5)_{n=-\infty} \cong -.52 \times 10^{-8}, \qquad R_5)_{n=0} \cong .30 \times 10^{-9}$$
$$A \text{ (absolute error)} \cong .30 \times 10^{-9}, \qquad R \text{ (relative error)} \cong .20 \times 10^{-8}.$$

The actual computation of (13) involves the subtraction of two large numbers with a corresponding loss of accuracy. Hence a transformation to a more satisfactory continued fraction was performed

$$\cos^* 1.3x = H_1 - 2z^2 + (H_3 + 320z^2)[z^2 + H_4 + H_5(z^2 + H_6)^{-1}]^{-1}$$

(14)

$H_1 =$	$10^2 \times .19477\ 14945\ 2366$	$H_5 =$	$10^4 \times .22874\ 43195\ 6870$
$H_3 =$	$-10^4 \times .32763\ 39951\ 6402$	$H_6 =$	$10^2 \times .24144\ 89469\ 4287$
$H_4 =$	$10^2 \times .82580\ 30199\ 5633$		

Check: 1) $\cos^*(0) = 1.00000\ 00005$

2) $\cos^*(1) = .54030\ 23025, \qquad \cos(1) = .54030\ 2306\cdots$

c) Tangent Approximation, 8-Digit Accuracy.

$$\tan\tfrac{1}{4}\pi x = (a_1T_1 + a_3T_3 + a_5T_5)(1 + b_2T_2)^{-1}$$

(15) $$= z[K_1 + K_2z^2 + K_3(z^2 + K_4)^{-1}] + \underbrace{(.165)10^{-6}\,z}_{\text{correction term}}$$

$-\tfrac{1}{4}\pi \leqq z = \tfrac{1}{4}\pi x \leqq \tfrac{1}{4}\pi,$ correction term added for $-.15 \leqq z \leqq .15$

$a_1 =$	$.86739\ 36410$	$K_1 =$	$.18717\ 82697\ 7$
$a_3 =$	$-10^{-1}\ .10096\ 94650$	$K_2 =$	$10^{-2}\ .41503\ 90625 = 2^{-7}\ (.100\ 01)_2$
$a_5 =$	$-10^{-4}\ .35876\ 64246$	$K_3 =$	$-10\ \ .20070\ 07228\ 1$
$b_2 =$	$-\ \ .14273\ 91684$	$K_4 =$	$-10\ \ .24691\ 85502\ 1$

A (absolute error) $\cong .842 \times 10^{-8}$

The relative error is reduced by the addition of the correction term.

Check: 1) $\tan^* z/z = 1.00000\,00001$ as $z \to 0$

2) $\tan^* (.1) = .10033\,4665,\qquad \tan (.1) = .10033\,467 \cdots$

3) $\tan^* (.7) = .84228\,83844,\qquad \tan (.7) = .84228\,838 \cdots$

d) Cotangent Approximation, 8-Digit Accuracy.

$$\cot^* \tfrac{1}{4}\pi x = (a_1T_1 + a_3T_3)(1 + b_2T_2 + b_4T_4)^{-1}$$

$$= 1/z[K_1 + K_2z^2 + K_3(z^2 + K_4)^{-1} \underbrace{- .526 \times 10^{-7}}_{\text{correction term}}]$$

(16)

$-\tfrac{1}{4}\pi \leqq z \leqq \tfrac{1}{4}\pi$, correction term added for $-.15 \leqq z \leqq .15$

$a_1 = .86369\,96360 \qquad K_1 = 10 \times .34180\,16667\,8$

$a_3 = -10^{-1} \times .13359\,27443\,6 \qquad K_2 = -.10156\,25 = -(.000\,110\,1)_2$

$b_2 = -.15019\,24454\,8 \qquad K_3 = 10^2 \times .25226\,53989\,66$

$b_4 = 10^{-3} \times .53281\,46284\,2 \qquad K_4 = -10^2 \times .10432\,74050\,83$

A (absolute error) $\cong .263 \times 10^{-8}$ to $.86 \times 10^{-8}$

Check: 1) $z \cdot \cot^* z = 1.0000000525 - .526 \times 10^{-7}$ as $z \to 0.$

2) $[\cot^* .1]^{-1} = .10033\,46678,\qquad \tan .1 = .10033\,467 \ldots$

3) $[\cot^* .15]^{-1} = .15113\,5221,\qquad \tan .15 = .15113\,522 \ldots$

e) Tangent Approximation, 10-Digit Accuracy.

$$\tan^* \tfrac{1}{4}\pi x = (a_1T_1 + a_3T_3 + a_5T_5)(1 + b_2T_2 + b_4T_4)^{-1}$$

$$= z\{(K_1 + K_2z^2)[z^2 + K_3 + K_4(z^2 + K_5)^{-1}]^{-1}\}$$

$-\tfrac{1}{4}\pi \leqq z = \tfrac{1}{4}\pi x \leqq \tfrac{1}{4}\pi$

$a_1 = .86130\,00805\,00276 \qquad K_1 = -10^1 \times .62993\,45787\,14378$

(17) $a_3 = -10^{-1} \times .15478\,46841\,31747 \qquad K_2 = .06738\,28125 = (.10001\,01)_2 \cdot 2^{-3}$

$a_5 = 10^{-4} \times .23254\,40722\,7 \qquad K_3 = -10^2 \times .17890\,72313\,8022$

$b_2 = -.15503\,39460\,54144 \qquad K_4 = 10^3 \times .11511\,65957\,03706$

$b_4 = 10^{-3} \times .87881\,25568\,77435 \qquad K_5 = -10^1 \times .99312\,26653\,90157$

It was again found desirable to transform to an equivalent approximation with smaller coefficients:

$$\tan^* z = z\{(H_1 + H_2z^2)[z^2 + H_3 + (H_4 - 16z^2)(z^2 + H_5)^{-1}]^{-1}\}$$

(18)

$H_1 = K_1,\qquad H_2 = K_2,\qquad H_3 = -10^1 \times .18907\,23138\,022$

$H_4 = 10^2 \times .43783\,03075\,87186 \qquad H_5 = K_5$

A (absolute error) $\cong .7 \times 10^{-11}$, $\qquad R$ (relative error) $\cong .83 \times 10^{-10}$

Check: 1) $\lim (\tan^* z)/z = 1.00000\,00000\,197$ as $z \to 0$.

2) $\tan^* \frac{1}{4}\pi x = (.78539\,81635\,2).4/\pi$, $\frac{1}{4}\pi = .78539\,8163 \cdots$

f) Sine Approximation, 10-Digit Accuracy.

$$\sin^* \tfrac{1}{4}\pi x = 2(a_1T_1 + a_3T_3 + a_5T_5)(1 + b_2T_2 + b_4T_4)^{-1}$$
$$= z\{K_1 + K_2[z^2 + K_3 + K_4(z^2 + K_5)^{-1}]^{-1}\}$$
$$= z\{H_1 + (H_2 + 6z^2)[z^2 + H_3 + H_4(z^2 + H_5)^{-1}]^{-1}\}$$
$$-\tfrac{1}{4}\pi \leqq z = \tfrac{1}{4}\pi x \leqq \tfrac{1}{4}\pi$$

(19)

$a_1 =$	$.36498\,44708\,84912$	$K_1 =$	$10^1 \times .71483\,02660\,86945$
$a_3 =$	$-10^{-2} \times .78595\,99360\,53327$	$K_2 =$	$-10^3 \times .80285\,00024\,48424$
$a_5 =$	$10^{-4} \times .30425\,20592\,02251$	$K_3 =$	$10^2 \times .55072\,65773\,70549$
$b_2 =$	$10^{-1} \times .10166\,05194\,0260$	$K_4 =$	$10^2 \times .12558\,99943\,5548$
$b_4 =$	$10^{-4} \times .21677\,07618\,0$	$K_5 =$	$10^2 \times .16632\,65254\,62696$
$H_1 =$	$10^1 \times .11483\,02660\,86945$	$H_4 =$	$10^4 \times .21114\,87369\,0673$
$H_2 =$	$-10^3 \times .37773\,44209\,59983$	$H_5 =$	$.85271\,33685\,84500$
$H_3 =$	$10^2 \times .70852\,59691\,47400$		

A (absolute error) $\cong .4356 \times 10^{-11}$, R (relative error) $\cong .863 \times 10^{-10}$

Check: 1) $\lim \sin^* z/z = .99999\,99999\,77$ as $z \to 0$

2) $\sin^* (.5) = .47942\,55386$, $\sin (.5) = .47942\,5539 \cdots$

g) Cosine Approximation, 10-Digit Accuracy.

$$\cos^* \tfrac{1}{4}\pi x = 2(a_0T_0 + a_2T_2 + a_4T_4 + a_6T_6)(1 + b_2T_2 + b_4T_4)^{-1}$$
$$= K_1 - 2z^2 + K_2[z^2 + K_3 + K_4(z^2 + K_5)^{-1}]^{-1}$$
$$= H_1 - 2z^2 + (H_2 + 338z^2)[z^2 + H_3 + H_4(z^2 + H_5)^{-1}]^{-1}$$
$$-\tfrac{1}{4}\pi \leqq z = \tfrac{1}{4}\pi x \leqq \tfrac{1}{4}\pi$$

(20)

$a_0 =$	$.42553\,53145\,92886$	$K_1 =$	$10^3 \times .33841\,65629\,20989$
$a_2 =$	$-10^{-1} \times .69950\,71904\,10770$	$K_2 =$	$-10^5 \times .29473\,89085\,26762$
$a_4 =$	$10^{-3} \times .68476\,48239\,11432$	$K_3 =$	$10^2 \times .57451\,41742\,44846$
$a_6 =$	$-10^{-5} \times .17046\,17065\,67264$	$K_4 =$	$10^4 \times .14616\,00034\,97481$
$b_2 =$	$10^{-2} \times .76660\,47537\,720$	$K_5 =$	$10^2 \times .48882\,57695\,11137$
$b_4 =$	$10^{-4} \times .11053\,68440\,0$		
$H_1 =$	$.41656\,29209\,89$	$H_4 =$	$10^4 \times .42283\,05642\,42266$
$H_2 =$	$10^4 \times .63340\,59841\,2301$	$H_5 =$	$.39331\,18492\,483$
$H_3 =$	$10^3 \times .10594\,06825\,2635$		

R (relative error) $\cong .5 \times 10^{-11}$

h) Logarithm Approximation, 8-Digit Accuracy.

$$\log_2{}^* f = Z[K_0 + K_1 \cdot Z^2 + K_2(K_3 + Z^2)^{-1}] - \tfrac{1}{2}, \qquad Z \equiv \frac{f - \frac{1}{2}\sqrt{2}}{f + \frac{1}{2}\sqrt{2}}$$

$$\tfrac{1}{2} \leqq f < 1$$

(21) $K_0 = 10^1 \times .18664\ 67623\ 69$ $\qquad K_2 = 10^1 \times .13545\ 03944\ 219$

$K_1 = .19531\ 25 \equiv .001\ 100\ 1)_2$ $\qquad K_3 = 10^1 \times .13293\ 49397\ 97$

A (absolute error) $\cong 2 \times 10^{-8}$

Several of the above approximations listed below have been incorporated in an Elementary Function Subroutine Package for the IBM computers 704 and 709.

Equation	*Machine*	*Share Distribution Number*	*Name*
(12), (14)	704, 709	510, 507	IB SIN1
(16)	704, 709	510, 507	IB TAN1
(17)	704, 709	510, 507	IB TAN2
(19), (20)	704, 709	571, 590	IB SIN2
(21)	709	665	IB LOG3

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