Some Relations and Values For the Generalized Riemann Zeta Function

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1. Introduction. In this paper we derive several relations among values of the generalized Riemann Zeta function $\zeta(s, a)$. We show that there is a "fundamental domain" for the variables s and a so that values outside this domain can be obtained from those inside by algebraic operations, using values of the Gamma and trigonometric functions.

We give a table of values of the Zeta function which were calculated using these relations to supplement the formulas commonly used for such calculations. Table 1 gives $\zeta(s, a)$ to seventeen decimal places for $a = \frac{1}{4}$ and $\frac{3}{4}$ and for $s = -\frac{61}{3}(\frac{1}{3})\frac{64}{3}$. The ordinary Riemann Zeta function is also given to seventeen decimal places for $s = 0(\frac{1}{3})\frac{64}{3}$ in Table 2. These tables are useful in diffraction theory [10], [11].

2. Relations among Zeta Functions. Consider the defining relation

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \tag{1}$$

which holds for $\mathrm{Re}(s) > 1$ and all values of a except zero or negative integers. (One usually assumes s complex but a real.) We first note that

$$\begin{aligned} \zeta(s, a + 1) &= \sum_{n=0}^{\infty} \frac{1}{(n + a + 1)^s} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} - \frac{1}{a^s} \\ &= \zeta(s, a) - a^{-s}. \end{aligned} \tag{2}$$

By analytic continuation, this relation holds for all s. Thus if $\zeta(s, a)$ is known for all s, but only for values of a satisfying $a_0 < a \leqq a_0 + 1$ for some a_0, then we can calculate $\zeta(s, a)$ for any value of a by algebraic operations. One usually specifies $a_0 = 0$, so that the "fundamental interval" is $0 < a \leqq 1$.

We now show that this fundamental interval in a need only be of length $\frac{1}{2}$. In the defining relation (1), let $a = (1/q) - b$ where q is an integer. Then

$$\begin{aligned} \zeta\left(s, \frac{1}{q} - b\right) &= q^s \sum_{n=0}^{\infty} \frac{1}{(nq + 1 - bq)^s} \\ &= q^s \left[\sum_{n=0}^{\infty} \frac{1}{(n + 1 - bq)^s} - \sum_{r=2}^{q} \sum_{n=0}^{\infty} \frac{1}{(nq + r - bq)^s}\right] \\ &= q^s \zeta(s, 1 - bq) - \sum_{r=2}^{q} \zeta\left(s, \frac{r}{q} - b\right). \end{aligned}$$

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Therefore

$$(3) \qquad \sum_{r=1}^{q} \zeta\left(s, \frac{r}{q} - b\right) = q^s \zeta(s, 1 - bq).$$

By analytic continuation, this result holds for all s. This relation was obtained by Powell [7] for the special case $s = \frac{1}{2}$.

Equation (3) yields many interesting and useful results. For example, let $b = \frac{1}{2} - c$. Then, for $q = 2$, we obtain

$$(4) \qquad \zeta(s, c) = 2^s \zeta(s, 2c) - \zeta(s, c + \tfrac{1}{2}).$$

For $c = \frac{1}{2}$ this yields the well known result

$$(5) \qquad \zeta(s, \tfrac{1}{2}) = (2^s - 1)\zeta(s),$$

where $\zeta(s)$ is the ordinary Riemann Zeta function defined by $\zeta(s, 1) = \zeta(s)$.

Now assume that $\zeta(s, a)$ is known for all s and for all a in the interval $\frac{1}{2} < a \leqq 1$. Then equation (4) shows that, if a_1 is such that $0 < a_1 \leqq \frac{1}{2}$, then $\zeta(s, a_1)$ can be expressed in terms of values assumed known. We merely apply relation (4) repeatedly as needed. An inductive proof can be easily obtained but will not be given here. As an example of the use of equation (4), consider the case $a_1 = \frac{1}{8}$. From equation (4),

$$(6) \qquad \zeta(s, \tfrac{1}{8}) = 2^s \zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{5}{8}).$$

The argument $\frac{1}{4}$ is not in the desired interval, so we use (4) to obtain

$$(7) \qquad \zeta(s, \tfrac{1}{4}) = 2^s \zeta(s, \tfrac{1}{2}) - \zeta(s, \tfrac{3}{4}).$$

Again, we do not know $\zeta(s, \frac{1}{2})$, so we use (4) to obtain

$$(8) \qquad \zeta(s, \tfrac{1}{2}) = 2^s \zeta(s, 1) - \zeta(s, 1) = (2^s - 1)\zeta(s).$$

Combining (6), (7) and (8), we have

$$\zeta(s, \tfrac{1}{8}) = 2^{2s}(2^s - 1)\zeta(s) - 2^s \zeta(s, \tfrac{3}{4}) - \zeta(s, \tfrac{5}{8}).$$

All the functions in the right member are assumed known, hence $\zeta(s, \frac{1}{8})$ is obtained by algebraic operations.

For special values of a, the more general relation (3) may be used more fruitfully than the special form (4).

We now see that $\frac{1}{2} < a \leqq 1$ is a fundamental interval for $\zeta(s, a)$. We next show that for rational values of a, we can obtain $\zeta(s, a)$ for $\operatorname{Re}(s) < \frac{1}{2}$ from values for $\operatorname{Re}(s) \geqq \frac{1}{2}$.

Consider the Hurwitz formula (see [1], page 269)

$$(9) \qquad \frac{(2\pi)^s}{2\Gamma(s)} \zeta(1 - s, a) = \cos \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^s} + \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi na}{n^s},$$

which is valid for $\operatorname{Re}(s) > 1$ and $0 < a \leqq 1$. Let $a = p/q$, where p and q are integers (which need not be relatively prime). Then

$$\frac{(2\pi)^s}{2\Gamma(s)}\zeta\left(1-s,\frac{p}{q}\right)=\cos\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\cos 2\pi np/q}{n^s}+\sin\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\sin 2\pi np/q}{n^s}$$

$$=\cos\frac{\pi s}{2}\sum_{r=0}^{q-1}\sum_{n=1}^{\infty}\frac{\cos 2\pi(qn-r)p/q}{(qn-r)^s}+\sin\frac{\pi s}{2}\sum_{r=0}^{q-1}\sum_{n=1}^{\infty}\frac{\sin 2\pi(qn-r)p/q}{(qn-r)^s}$$

$$=\frac{1}{q^s}\cos\frac{\pi s}{2}\sum_{r=0}^{q-1}\sum_{n=1}^{\infty}\frac{\cos 2\pi rp/q}{(n-r/q)^s}-\frac{1}{q^s}\sin\frac{\pi s}{2}\sum_{r=0}^{q-1}\sum_{n=1}^{\infty}\frac{\sin 2\pi rp/q}{(n-r/q)^s}$$

$$=\frac{1}{q^s}\sum_{r=0}^{q-1}\left[\cos\frac{\pi s}{2}\cos\frac{2\pi rp}{q}-\sin\frac{\pi s}{2}\sin\frac{2\pi rp}{q}\right]\sum_{n=0}^{\infty}\frac{1}{(n+1-r/q)^s}.$$

Replacing r by $q-r$ and simplifying,

$$\zeta(1-s,p/q)=\frac{2\Gamma(s)}{(2\pi q)^s}\sum_{r=1}^{q}\cos\pi\left(\frac{s}{2}-\frac{2rp}{q}\right)\zeta(s,r/q). \tag{10}$$

By analytic continuation, this result holds for all s. Thus, if $\zeta(s, a)$ is known for $\mathrm{Re}(s) \geqq \frac{1}{2}$ and all rational a, we can compute $\zeta(s, a)$ for all s and all rational a, $0 < a \leqq 1$, from equation (10). Conversely, if we know $\zeta(s, a)$ for all rational a, $0 < a \leqq 1$, and for $\mathrm{Re}(s) \leqq \frac{1}{2}$, we can compute $\zeta(s, a)$ from equation (10) for; all s except for integer values. The right hand side of equation (10) contains $\Gamma(s)$, which is infinite for s a negative integer. Hence equation (10) cannot, in general, be used directly to yield $\zeta(s, a)$ for s a positive integer. However equations (2), (4), and (10) enable us to find $\zeta(s, a)$ for all s and all rational a from values of $\zeta(s, a)$ for $\mathrm{Re}(s) \geqq \frac{1}{2}$ and $\frac{1}{2} < a \leqq 1$.

Let us now consider some special cases of equation (10). For $p = 1$ and $q = 2$, we again obtain equation (5). For $p = q = 1$, we obtain the Riemann relation

$$\zeta(1-s)=\frac{2\Gamma(s)}{(2\pi)^s}\cos\frac{\pi s}{2}\zeta(s). \tag{11}$$

For the general case $p = q$, we obtain, by using relation (11),

$$\sum_{r=1}^{q-1}\zeta(s,r/q)=(q^s-1)\zeta(s) \tag{12}$$

which also follows from equation (3). If we also write equation (12) with s replaced by $1 - s$, then the resulting equation can be combined with equations (11) and (12) to yield

$$\begin{aligned}\pi^{-s/2}\Gamma(s/2)(q^s-1)\sum_{r=1}^{q-1}\zeta(s,r/q)\\ =\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)(q^{1-s}-1)\sum_{r=1}^{q-1}\zeta(1-s,r/q).\end{aligned} \tag{13}$$

That is, the members of this equation are invariant under replacement of s by $1 - s$. Hence we may regard equation (13) as a generalization of the Riemann relation (11), which is often written as

$$\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s).$$

Next substitute $q = 4$ and $p = 1$ and 3 in equation (10). In this case, we obtain, using equations (5) and (11)

$$(14) \quad \zeta(1-s, \tfrac{1}{2} \pm \tfrac{1}{4}) = \frac{2-2^s}{4^s}\zeta\,(1-s) \pm \frac{2\Gamma(s)}{(8\pi)^s}\sin\frac{\pi s}{2}\,[\zeta(s, \tfrac{3}{4}) - \zeta(s, \tfrac{1}{4})].$$

Subtracting this equation, using the lower sign from the equation using the upper sign, we obtain

$$(15) \qquad \zeta(1-s, \tfrac{3}{4}) - \zeta(1-s, \tfrac{1}{4}) = \frac{4\Gamma(s)}{(8\pi)^s}\sin\frac{\pi s}{2}\,[\zeta(s, \tfrac{3}{4}) - \zeta(s, \tfrac{1}{4})].$$

Now the Dirichlet L-function is defined to be

$$L(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

for $\mathrm{Re}(s) > 1$. This can be expressed as

$$L(s) = 4^{-s}[\zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{3}{4})].$$

Hence equation (15) can be rewritten as

$$L(1-s) = (2/\pi)^s\Gamma(s)\sin\frac{\pi s}{2}\,L(s).$$

This relation is derived (but misprinted) by Titchmarsh (see [6], page 66).

For m equal to a positive integer,

$$(16) \qquad \zeta(-m, a) = -\frac{B_{m+1}(a)}{(m+1)},$$

where $B_m(a)$ is the mth Bernoulli polynomial in a. (See [1], page 267.) Hence, letting $s \to -2m$ in equation (14), and using the fact that $B_m(1-a) = (-1)^m B_m(a)$, we obtain

$$\zeta(2m+1, \tfrac{1}{2} \pm \tfrac{1}{4}) = 2^{2m}(2^{2m+1} - 1)\zeta(2m+1) \pm \frac{2^{6m+1}\pi^{2m+1}}{(2m+1)!}B_{2m+1}(\tfrac{1}{4}).$$

From these equations, we find that

$$L(2m+1) = -\frac{(2\pi)^{2m+1}}{2(2m+1)!}B_{2m+1}(\tfrac{1}{4}).$$

Many other interesting results can be obtained by looking at special cases of equation (10).

The method used to derive equations (3) and (10) can also be applied to the more general function

$$\zeta(s, a, z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

discussed by Mitchell [8], Lerch [9] and others.

3. Calculation of the Table. In the discussion above we showed, by using equation (10), that we could find $\zeta(s, a)$ for the $\mathrm{Re}(s) \leqq \frac{1}{2}$ and rational a, $0 < a \leqq 1$, provided we know $\zeta(s, a)$ for $\mathrm{Re}(s) > \frac{1}{2}$ and rational a, $0 < a \leqq 1$. The equation

$$\zeta(s, a) = \sum_{n=0}^{N-1} \frac{1}{(n+a)^s} - \frac{(N+a)^{1-s}}{1-s} + \frac{(N+a)^{-s}}{2} - \sum_{n=1}^{\infty} \frac{B_n}{(2n)!} \frac{\Gamma(1-s)}{\Gamma(2-s-2n)} (N+a)^{-s-(2n-1)}, \tag{17}$$

where $B_1 = \frac{1}{6}$, $B_2 = -\frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = -\frac{1}{30}$, $B_5 = \frac{5}{66}$, $\cdots$ are Bernoulli numbers, is a representation of $\zeta(s, a)$ for $\mathrm{Re}(s) > 0$, $s \neq 1$, and $0 < a \leqq 1$. We use this asymptotic series to calculate $\zeta(s, \frac{1}{4})$, $\zeta(s, \frac{3}{4})$, and $\zeta(s, 1) = \zeta(s)$ for $s = \frac{2}{3}(\frac{1}{3})\frac{64}{3}$. It is not necessary to use this expression to calculate $\zeta(s, \frac{1}{2})$ because of equation (8). With these values of $\zeta(s, \frac{1}{4})$, $\zeta(s, \frac{3}{4})$, and $\zeta(s, 1)$ for $s = \frac{2}{3}(\frac{1}{3})\frac{64}{3}$, we calculate $\zeta(s, \frac{1}{4})$ and $\zeta(s, \frac{3}{4})$, using (14), for $s = -\frac{61}{3}(\frac{1}{3})\frac{1}{3}$ with the exception $s = 0$. For $s = 0$ we use the identity $\zeta(0, a) = \frac{1}{2} - a$. The Gamma function, $\Gamma(s)$, appearing on the right side of equation (10) is calculated by making use of the asymptotic series (see [1] Section 12.33)

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log (2\pi) + \sum_{r=1}^{\infty} \frac{(-1)^{r-1} B_r}{2r(2r-1)s^{2r-1}}$$

where B_r are the Bernoulli numbers. This expression holds for $|\arg s| \leqq \frac{1}{2}\pi - \Delta$ and $0 < \Delta < \frac{1}{4}\pi$. Using this relation in conjunction with the equation $\Gamma(s+1) = s\Gamma(s)$, we obtain the desired values of the Gamma function.

Equation (17), for $a = 1$, is equivalent to the expression used by Gram [2] and by Haselgrove [3] in preparing their tables of the ordinary Riemann Zeta function.

4. Methods of Checking the Table Values. Equation (16) was used to check the table entries for negative integer values of s, while Hurwitz's formula (9) was used to check the entries for $s = -\frac{61}{3}(\frac{1}{3}) - \frac{28}{3}$. For positive integer values of s, the values of $\zeta(s)$ agree exactly with the table in [5], page xxv. Our table was not checked for other values of s, since no other simple method seemed to present itself. Note, however, the table for negative values of s was calculated using the entries for positive values of s. Hence, it seems safe to assume that the table entries for positive values of s are correct to at least as many digits as the table entries for corresponding negative values of s.

5. Comments Regarding the Calculation of the Table. The table shown below was calculated using a precision of seventy binary digits, which is approximately equivalent to twenty-one decimal digits.

It was impossible to enter most of the values of s exactly in the computer, since 1/3 does not have a finite binary representation. Therefore, we should consider the errors introduced in our table from using a truncated binary representation. We were able to show, by using a truncated Taylor series, that the errors so introduced did not affect the accuracy of the table. The various calculations involved in showing the insignificance of these errors were not included in this paper, since they were quite straightforward.

6. The Table of Values. The tabular data have been listed to seventeen significant decimal digits. The decimal point is located immediately to the left of the left-most digit of each entry, while the two-digit integer at the right denotes the exponent of the power of ten by which the entry is to be multiplied.

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TABLE 1

s	$\zeta(s, \frac{1}{4})$		$\zeta(s, \frac{3}{4})$	
64/3	69814 63658 33156 77	13	46276 41642 09957 54	03
21	43980 46511 10400 92	13	42044 91481 42985 79	03
62/3	27705 95688 87832 77	13	38200 34049 99724 72	03
61/3	17453 65914 58290 05	13	34707 31296 92456 16	03
20	10995 11627 77601 15	13	31533 68689 82531 57	03
59/3	69264 89222 19591 87	12	28650 25632 40314 70	03
58/3	43634 14786 45735 82	12	26030 48587 06422 96	03
19	27487 79069 44014 41	12	23650 26655 19854 06	03
56/3	17316 22305 54910 39	12	21487 69390 40321 69	03
55/3	10908 53696 61447 33	12	19522 86640 47054 77	03
18	68719 47673 60180 15	11	17737 70232 63351 39	03
53/3	43290 55763 87431 22	11	16115 77333 52502 14	03
52/3	27271 34241 53785 57	11	14642 15330 71921 73	03
17	17179 86918 40225 19	11	13303 28096 72635 09	03
50/3	10822 63940 97051 87	11	12086 83509 03545 36	03
49/3	68178 35603 86554 48	10	10981 62111 35873 48	03
16	42949 67296 02814 98	10	99774 68117 34460 20	02
47/3	27056 59852 45055 63	10	90651 35226 87845 72	02
46/3	17044 58900 99251 96	10	82362 26573 20047 47	02
15	10737 41824 03518 96	10	74831 14030 73469 44	02
44/3	67641 49631 42966 14	09	67988 67020 79914 81	02
43/3	42611 47252 80800 17	09	61771 88735 02511 35	02
14	26843 54560 43992 27	09	56123 58191 92793 72	02
41/3	16910 37408 23656 51	09	50991 77593 55328 03	02
40/3	10652 86813 61045 94	09	46329 24497 93384 86	02
13	67108 86405 50022 12	08	42093 08367 45511 12	02
38/3	42275 93525 33217 14	08	38244 31093 55480 72	02
37/3	26632 17039 13370 57	08	34747 51134 84446 15	02
12	16777 21606 87796 31	08	31570 50939 08321 59	02
35/3	10568 98387 26213 03	08	28684 07349 76745 21	02
34/3	66580 42661 71997 90	07	26061 64725 71060 80	02
11	41943 04086 03548 00	07	23679 10527 14066 46	02
32/3	26422 46042 34022 87	07	21514 53144 78372 37	02
31/3	16645 10745 38096 69	07	19548 01769 36192 33	02
10	10485 76107 68311 48	07	17761 48118 13370 04	02
29/3	66056 16034 78926 59	06	16138 49852 76111 40	02
28/3	41612 77864 98520 10	06	14664 15539 12355 32	02
9	26214 41349 21724 08	06	13324 91014 94593 48	02
26/3	16514 05173 61085 33	06	12108 47045 54080 94	02
25/3	10403 20723 10878 44	06	11003 68161 90407 28	02
8	65536 16938 67090 17	05	10000 42589 27890 58	02
23/3	41285 27582 34437 32	05	90895 31887 04304 92	01
22/3	26008 17634 03920 72	05	82626 93498 64416 39	01
7	16384 21345 51995 72	05	75123 97920 96579 80	01
20/3	10321 50411 38500 09	05	68318 62538 56294 03	01
19/3	65022 44664 70976 70	04	62149 80828 62956 03	01
6	40962 70948 06401 08	04	56562 77857 28816 43	01
17/3	25806 12469 70992 37	04	51508 76680 73079 38	01
16/3	16258 18668 37913 66	04	46944 78141 49920 25	01
5	10243 48974 52658 06	04	42833 58575 64239 47	01
14/3	64546 14107 47051 67	03	39143 93590 59400 43	01
13/3	40679 42509 60218 28	03	35851 22960 35115 28	01

TABLE 1—*Continued*

s	$\zeta(s, \frac{1}{4})$		$\zeta(s, \frac{3}{4})$	
4	25646 36906 68198 07	03	32938 85422 47510 00	01
11/3	16178 62720 13961 71	03	30400 81320 64265 08	01
10/3	10217 48139 83634 30	03	28246 87653 98801 74	01
3	64663 86996 87684 60	02	26513 16608 16881 98	01
8/3	41092 64168 75447 99	02	25285 74853 60394 04	01
7/3	26334 77863 02380 64	02	24759 96055 02275 82	01
2	17197 32915 45071 11	02	25418 79647 67160 65	01
5/3	11787 30284 47967 72	02	28747 02273 10830 06	01
4/3	95676 33344 56813 25	01	42231 01605 70504 20	01
1	∞		∞	
2/3	−24411 86144 96889 78	00	−20381 06131 82465 56	01
1/3	33101 39009 27282 69	00	−64976 99170 75290 43	00
0	25000 00000 00000 00	00	−25000 00000 00000 00	00
−1/3	13499 19957 79665 53	00	−89579 84483 88113 98	−01
−2/3	55399 96817 77458 36	−01	−19221 97815 54288 52	−01
−1	10416 66666 66666 67	−01	10416 66666 66666 67	−01
−4/3	−10558 77278 36124 00	−01	20147 85800 02319 75	−01
−5/3	−16992 51822 29812 52	−01	20093 86425 54835 96	−01
−2	−15625 00000 00000 00	−01	15625 00000 00000 00	−01
−7/3	−10834 45058 95508 42	−01	97268 16010 89067 10	−02
−8/3	−52815 71878 70004 44	−02	40704 00253 32719 07	−02
−3	−45572 91666 66666 67	−03	−45572 91666 66666 67	−03
−10/3	29301 12394 80509 84	−02	−34597 28641 05474 40	−02
−11/3	46698 19500 95810 35	−02	−48805 58903 88315 47	−02
−4	48828 12500 00000 00	−02	−48828 12500 00000 00	−02
−13/3	38899 36684 63823 16	−02	−37822 44498 22381 09	−02
−14/3	21239 95040 97140 41	−02	−19861 15110 11172 00	−02
−5	60066 34424 60317 46	−04	60066 34424 60317 46	−04
−16/3	−18432 75029 83014 06	−02	19233 18930 79122 45	−02
−17/3	−31960 86810 98371 30	−02	32323 03687 52690 05	−02
−6	−37231 44531 25000 00	−02	37231 44531 24500 00	−02
−19/3	−33017 11705 70240 87	−02	32782 87245 91040 61	−02
−20/3	−19865 27145 97911 51	−02	19530 95592 98213 26	−02
−7	−16148 88509 11458 33	−04	−16148 88509 11458 33	−04
−22/3	22055 37817 57697 42	−02	−22292 83934 16781 00	−02
−23/3	41577 48345 17227 95	−02	−41695 51768 48320 96	−02
−8	52833 55712 89062 50	−02	−52833 55712 89062 50	−02
−25/3	50987 50282 80912 53	−02	−50896 45120 98702 91	−02
−26/3	33229 98874 70615 11	−02	−33088 82387 94668 38	−02
−9	73837 51146 72111 74	−05	73837 51146 72111 74	−05
−28/3	−43631 40309 49183 10	−02	43748 62335 98758 29	−02
−29/3	−88331 79840 49455 97	−02	88394 53105 86276 23	−02
−10	−12045 14503 47900 39	−01	12045 14503 47900 39	−01
−31/3	−12448 12326 45933 15	−01	12442 55605 16962 73	−01
−32/3	−86654 89190 70478 86	−02	86562 63379 78600 91	−02
−11	−51470 93963 85593 44	−05	−51470 93963 85593 44	−05
−34/3	−12956 24917 98495 48	−01	−12964 94737 89903 23	−01
−35/3	27851 69352 45565 33	−01	−27856 63944 24212 63	−01
−12	40274 33693 40896 61	−01	−40274 33693 40896 61	−01
−37/3	44066 74225 89134 15	−01	−44061 81300 66818 43	−01
−38/3	32424 26287 38385 85	−01	−32415 62726 12390 43	−01
−13	50856 42139 11692 30	−05	50856 42139 11692 30	−05
−40/3	−53981 95160 35609 58	−01	53991 01083 71719 25	−01

TABLE 1—*Continued*

s	$\zeta(s, \frac{1}{4})$		$\zeta(s, \frac{3}{4})$	
−41/3	−12213 66720 37044 43	00	12214 20945 65951 94	00
−14	−18566 93821 02787 49	00	18566 93821 02787 49	00
−43/3	−21331 38708 22892 45	00	21330 79048 86111 17	00
−44/3	−16461 18663 02940 22	00	16460 09042 37713 87	00
−15	−67634 01438 99342 01	−05	−67634 01438 99342 01	−05
−46/3	30053 84266 68572 82	00	−30055 10353 95330 80	00
−47/3	71088 87471 07870 00	00	−71089 66375 73979 33	00
−16	11287 34563 53554 50	01	−11287 34563 53554 50	01
−49/3	13532 13201 24995 85	01	−13532 03739 35347 35	01
−50/3	10887 19301 23970 26	01	−10887 01175 14453 29	01
−17	11649 82235 12560 03	−04	11649 82235 12560 03	−04
−52/3	−21552 81617 38443 47	01	21553 04222 79286 09	01
−53/3	−53020 89805 52583 77	01	53021 04518 12564 35	01
−18	−87489 01292 85995 50	01	87489 01292 85995 50	01
−55/3	−10892 53279 53442 03	02	10892 51375 45982 56	02
−56/3	−90944 16979 46386 51	01	90943 79125 83854 77	01
−19	−25230 56188 58262 09	−04	−25230 56188 58262 09	−04
−58/3	19350 58818 71838 18	02	−19350 63892 61704 53	02
−59/3	49304 35983 54058 91	02	−49304 39403 87055 58	02
−20	84212 65834 78432 17	02	−84212 65834 78432 17	02
−61/3	10846 36397 60161 63	03	−10846 35923 59976 00	03

TABLE 2

s	$\zeta(s)$	s	$\zeta(s)$
64/3	10000 00378 53233 25 01	31/3	10007 87512 59735 22 01
21	10000 00476 93298 68 01	10	10009 94575 12781 81 01
62/3	10000 00600 91541 53 01	29/3	10012 56544 89496 33 01
61/3	10000 00757 13123 99 01	28/3	10015 88198 62306 52 01
20	10000 00953 96203 39 01	9	10020 08392 82608 22 01
59/3	10000 01201 96944 13 01	26/3	10025 41240 15797 39 01
58/3	10000 01514 46249 87 01	25/3	10032 17649 68406 42 01
19	10000 01908 21271 66 01	8	10040 77356 19794 43 01
56/3	10000 02404 35545 91 01	23/3	10051 71613 23892 26 01
55/3	10000 03029 52623 90 01	22/3	10065 66797 08110 32 01
18	10000 03817 29326 50 01	7	10083 49277 38192 28 01
53/3	10000 04809 96364 94 01	20/3	10106 32075 32048 38 01
52/3	10000 06060 86098 24 01	19/3	10135 64087 21870 02 01
17	10000 07637 19763 79 01	6	10173 43061 98444 91 01
50/3	10000 09623 69761 94 01	17/3	10222 34193 45949 71 01
49/3	10000 12127 16680 42 01	16/3	10285 97325 52868 25 01
16	10000 15282 25940 87 01	5	10369 27755 14336 99 01
47/3	10000 19258 75533 22 01	14/3	10479 19229 28351 53 01
46/3	10000 24270 74650 59 01	13/3	10625 64589 86109 19 01
15	10000 30588 23630 70 01	4	10823 23233 71113 82 01
44/3	10000 38551 79067 62 01	11/3	11094 13692 59748 37 01
43/3	10000 48591 05077 17 01	10/3	11473 56236 88274 29 01
14	10000 61248 13505 87 01	3	12020 56903 15959 43 01
41/3	10000 77207 23697 67 01	8/3	12841 90540 23974 21 01
40/3	10000 97332 08000 25 01	7/3	14151 55609 44598 30 01
13	10001 22713 34757 85 01	2	16449 34066 84822 64 01
38/3	10001 54728 79053 14 01	5/3	21235 22968 85758 35 01
37/3	10001 95119 46821 60 01	4/3	36009 37750 45886 24 01
12	10002 46086 55330 80 01	1	∞
35/3	10003 10414 39287 00 01	2/3	−24475 80736 23365 82 01
34/3	10003 91627 16273 87 01	1/3	−97336 02483 50782 72 01
11	10004 94188 60411 95 01	0	−50000 00000 00000 00 00
32/3	10006 23757 16251 28 01		