

Some Relations and Values For the Generalized Riemann Zeta Function

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1. Introduction. In this paper we derive several relations among values of the generalized Riemann Zeta function $\zeta(s, a)$. We show that there is a "fundamental domain" for the variables s and a so that values outside this domain can be obtained from those inside by algebraic operations, using values of the Gamma and trigonometric functions.

We give a table of values of the Zeta function which were calculated using these relations to supplement the formulas commonly used for such calculations. Table 1 gives $\zeta(s, a)$ to seventeen decimal places for $a = \frac{1}{4}$ and $\frac{3}{4}$ and for $s = -\frac{a-1}{3}(\frac{1}{3})^{\frac{a-4}{3}}$. The ordinary Riemann Zeta function is also given to seventeen decimal places for $s = 0(\frac{1}{3})^{\frac{a-4}{3}}$ in Table 2. These tables are useful in diffraction theory [10], [11].

2. Relations among Zeta Functions. Consider the defining relation

$$(1) \quad \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

which holds for $\text{Re}(s) > 1$ and all values of a except zero or negative integers. (One usually assumes s complex but a real.) We first note that

$$(2) \quad \begin{aligned} \zeta(s, a+1) &= \sum_{n=0}^{\infty} \frac{1}{(n+a+1)^s} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} - \frac{1}{a^s} \\ &= \zeta(s, a) - a^{-s}. \end{aligned}$$

By analytic continuation, this relation holds for all s . Thus if $\zeta(s, a)$ is known for all s , but only for values of a satisfying $a_0 < a \leq a_0 + 1$ for some a_0 , then we can calculate $\zeta(s, a)$ for any value of a by algebraic operations. One usually specifies $a_0 = 0$, so that the "fundamental interval" is $0 < a \leq 1$.

We now show that this fundamental interval in a need only be of length $\frac{1}{2}$. In the defining relation (1), let $a = (1/q) - b$ where q is an integer. Then

$$\begin{aligned} \zeta\left(s, \frac{1}{q} - b\right) &= q^s \sum_{n=0}^{\infty} \frac{1}{(nq + 1 - bq)^s} \\ &= q^s \left[\sum_{n=0}^{\infty} \frac{1}{(n+1-bq)^s} - \sum_{r=2}^q \sum_{n=0}^{\infty} \frac{1}{(nq+r-bq)^s} \right] \\ &= q^s \zeta(s, 1-bq) - \sum_{r=2}^q \zeta\left(s, \frac{r}{q} - b\right). \end{aligned}$$

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Therefore

$$(3) \quad \sum_{r=1}^q \zeta\left(s, \frac{r}{q} - b\right) = q^s \zeta(s, 1 - bq).$$

By analytic continuation, this result holds for all s . This relation was obtained by Powell [7] for the special case $s = \frac{1}{2}$.

Equation (3) yields many interesting and useful results. For example, let $b = \frac{1}{2} - c$. Then, for $q = 2$, we obtain

$$(4) \quad \zeta(s, c) = 2^s \zeta(s, 2c) - \zeta(s, c + \frac{1}{2}).$$

For $c = \frac{1}{2}$ this yields the well known result

$$(5) \quad \zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s),$$

where $\zeta(s)$ is the ordinary Riemann Zeta function defined by $\zeta(s, 1) = \zeta(s)$.

Now assume that $\zeta(s, a)$ is known for all s and for all a in the interval $\frac{1}{2} < a \leq 1$. Then equation (4) shows that, if a_1 is such that $0 < a_1 \leq \frac{1}{2}$, then $\zeta(s, a_1)$ can be expressed in terms of values assumed known. We merely apply relation (4) repeatedly as needed. An inductive proof can be easily obtained but will not be given here. As an example of the use of equation (4), consider the case $a_1 = \frac{1}{8}$. From equation (4),

$$(6) \quad \zeta(s, \frac{1}{8}) = 2^s \zeta(s, \frac{1}{4}) - \zeta(s, \frac{5}{8}).$$

The argument $\frac{1}{4}$ is not in the desired interval, so we use (4) to obtain

$$(7) \quad \zeta(s, \frac{1}{4}) = 2^s \zeta(s, \frac{1}{2}) - \zeta(s, \frac{3}{4}).$$

Again, we do not know $\zeta(s, \frac{1}{2})$, so we use (4) to obtain

$$(8) \quad \zeta(s, \frac{1}{2}) = 2^s \zeta(s, 1) - \zeta(s, 1) = (2^s - 1)\zeta(s).$$

Combining (6), (7) and (8), we have

$$\zeta(s, \frac{1}{8}) = 2^{2s}(2^s - 1)\zeta(s) - 2^s \zeta(s, \frac{3}{4}) - \zeta(s, \frac{5}{8}).$$

All the functions in the right member are assumed known, hence $\zeta(s, \frac{1}{8})$ is obtained by algebraic operations.

For special values of a , the more general relation (3) may be used more fruitfully than the special form (4).

We now see that $\frac{1}{2} < a \leq 1$ is a fundamental interval for $\zeta(s, a)$. We next show that for rational values of a , we can obtain $\zeta(s, a)$ for $\text{Re}(s) < \frac{1}{2}$ from values for $\text{Re}(s) \geq \frac{1}{2}$.

Consider the Hurwitz formula (see [1], page 269)

$$(9) \quad \frac{(2\pi)^s}{2\Gamma(s)} \zeta(1-s, a) = \cos \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi n a}{n^s} + \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi n a}{n^s},$$

which is valid for $\text{Re}(s) > 1$ and $0 < a \leq 1$. Let $a = p/q$, where p and q are in-

tegers (which need not be relatively prime). Then

$$\begin{aligned} \frac{(2\pi)^s}{2\Gamma(s)} \zeta\left(1 - s, \frac{p}{q}\right) &= \cos \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi n p/q}{n^s} + \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi n p/q}{n^s} \\ &= \cos \frac{\pi s}{2} \sum_{r=0}^{q-1} \sum_{n=1}^{\infty} \frac{\cos 2\pi (qn - r)p/q}{(qn - r)^s} + \sin \frac{\pi s}{2} \sum_{r=0}^{q-1} \sum_{n=1}^{\infty} \frac{\sin 2\pi (qn - r)p/q}{(qn - r)^s} \\ &= \frac{1}{q^s} \cos \frac{\pi s}{2} \sum_{r=0}^{q-1} \sum_{n=1}^{\infty} \frac{\cos 2\pi r p/q}{(n - r/q)^s} - \frac{1}{q^s} \sin \frac{\pi s}{2} \sum_{r=0}^{q-1} \sum_{n=1}^{\infty} \frac{\sin 2\pi r p/q}{(n - r/q)^s} \\ &= \frac{1}{q^s} \sum_{r=0}^{q-1} \left[\cos \frac{\pi s}{2} \cos \frac{2\pi r p}{q} - \sin \frac{\pi s}{2} \sin \frac{2\pi r p}{q} \right] \sum_{n=0}^{\infty} \frac{1}{(n + 1 - r/q)^s}. \end{aligned}$$

Replacing r by $q - r$ and simplifying,

$$(10) \quad \zeta(1 - s, p/q) = \frac{2\Gamma(s)}{(2\pi q)^s} \sum_{r=1}^q \cos \pi \left(\frac{s}{2} - \frac{2rp}{q}\right) \zeta(s, r/q).$$

By analytic continuation, this result holds for all s . Thus, if $\zeta(s, a)$ is known for $\text{Re}(s) \geq \frac{1}{2}$ and all rational a , we can compute $\zeta(s, a)$ for all s and all rational a , $0 < a \leq 1$, from equation (10). Conversely, if we know $\zeta(s, a)$ for all rational a , $0 < a \leq 1$, and for $\text{Re}(s) \leq \frac{1}{2}$, we can compute $\zeta(s, a)$ from equation (10) for all s except for integer values. The right hand side of equation (10) contains $\Gamma(s)$, which is infinite for s a negative integer. Hence equation (10) cannot, in general, be used directly to yield $\zeta(s, a)$ for s a positive integer. However equations (2), (4), and (10) enable us to find $\zeta(s, a)$ for all s and all rational a from values of $\zeta(s, a)$ for $\text{Re}(s) \geq \frac{1}{2}$ and $\frac{1}{2} < a \leq 1$.

Let us now consider some special cases of equation (10). For $p = 1$ and $q = 2$, we again obtain equation (5). For $p = q = 1$, we obtain the Riemann relation

$$(11) \quad \zeta(1 - s) = \frac{2\Gamma(s)}{(2\pi)^s} \cos \frac{\pi s}{2} \zeta(s).$$

For the general case $p = q$, we obtain, by using relation (11),

$$(12) \quad \sum_{r=1}^{q-1} \zeta(s, r/q) = (q^s - 1)\zeta(s)$$

which also follows from equation (3). If we also write equation (12) with s replaced by $1 - s$, then the resulting equation can be combined with equations (11) and (12) to yield

$$(13) \quad \begin{aligned} \pi^{-s/2} \Gamma(s/2) (q^s - 1) \sum_{r=1}^{q-1} \zeta(s, r/q) \\ = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) (q^{1-s} - 1) \sum_{r=1}^{q-1} \zeta(1 - s, r/q). \end{aligned}$$

That is, the members of this equation are invariant under replacement of s by $1 - s$. Hence we may regard equation (13) as a generalization of the Riemann

relation (11), which is often written as

$$\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s).$$

Next substitute $q = 4$ and $p = 1$ and 3 in equation (10). In this case, we obtain, using equations (5) and (11)

$$(14) \quad \zeta\left(1-s, \frac{1}{2} \pm \frac{1}{4}\right) = \frac{2-2^s}{4^s}\zeta(1-s) \pm \frac{2\Gamma(s)}{(8\pi)^s} \sin \frac{\pi s}{2} [\zeta\left(s, \frac{3}{4}\right) - \zeta\left(s, \frac{1}{4}\right)].$$

Subtracting this equation, using the lower sign from the equation using the upper sign, we obtain

$$(15) \quad \zeta\left(1-s, \frac{3}{4}\right) - \zeta\left(1-s, \frac{1}{4}\right) = \frac{4\Gamma(s)}{(8\pi)^s} \sin \frac{\pi s}{2} [\zeta\left(s, \frac{3}{4}\right) - \zeta\left(s, \frac{1}{4}\right)].$$

Now the Dirichlet L -function is defined to be

$$L(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

for $\text{Re}(s) > 1$. This can be expressed as

$$L(s) = 4^{-s}[\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right)].$$

Hence equation (15) can be rewritten as

$$L(1-s) = (2/\pi)^s \Gamma(s) \sin \frac{\pi s}{2} L(s).$$

This relation is derived (but misprinted) by Titchmarsh (see [6], page 66).

For m equal to a positive integer,

$$(16) \quad \zeta(-m, a) = -\frac{B_{m+1}(a)}{(m+1)},$$

where $B_m(a)$ is the m th Bernoulli polynomial in a . (See [1], page 267.) Hence, letting $s \rightarrow -2m$ in equation (14), and using the fact that $B_m(1-a) = (-1)^m B_m(a)$, we obtain

$$\zeta\left(2m+1, \frac{1}{2} \pm \frac{1}{4}\right) = 2^{2m}(2^{2m+1}-1)\zeta(2m+1) \pm \frac{2^{6m+1}\pi^{2m+1}}{(2m+1)!} B_{2m+1}\left(\frac{1}{4}\right).$$

From these equations, we find that

$$L(2m+1) = -\frac{(2\pi)^{2m+1}}{2(2m+1)!} B_{2m+1}\left(\frac{1}{4}\right).$$

Many other interesting results can be obtained by looking at special cases of equation (10).

The method used to derive equations (3) and (10) can also be applied to the more general function

$$\zeta(s, a, z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

discussed by Mitchell [8], Lerch [9] and others.

3. Calculation of the Table. In the discussion above we showed, by using equation (10), that we could find $\zeta(s, a)$ for the $\text{Re}(s) \leq \frac{1}{2}$ and rational $a, 0 < a \leq 1$, provided we know $\zeta(s, a)$ for $\text{Re}(s) > \frac{1}{2}$ and rational $a, 0 < a \leq 1$. The equation

$$(17) \quad \zeta(s, a) = \sum_{n=0}^{N-1} \frac{1}{(n+a)^s} - \frac{(N+a)^{1-s}}{1-s} + \frac{(N+a)^{-s}}{2} - \sum_{n=1}^{\infty} \frac{B_n}{(2n)!} \frac{\Gamma(1-s)}{\Gamma(2-s-2n)} (N+a)^{-s-(2n-1)},$$

where $B_1 = \frac{1}{6}, B_2 = -\frac{1}{30}, B_3 = \frac{1}{42}, B_4 = -\frac{1}{30}, B_5 = \frac{5}{66}, \dots$ are Bernoulli numbers, is a representation of $\zeta(s, a)$ for $\text{Re}(s) > 0, s \neq 1$, and $0 < a \leq 1$. We use this asymptotic series to calculate $\zeta(s, \frac{1}{4}), \zeta(s, \frac{3}{4})$, and $\zeta(s, 1) = \zeta(s)$ for $s = \frac{2}{3}(\frac{1}{3})^{\frac{2}{3}}$. It is not necessary to use this expression to calculate $\zeta(s, \frac{1}{2})$ because of equation (8). With these values of $\zeta(s, \frac{1}{4}), \zeta(s, \frac{3}{4})$, and $\zeta(s, 1)$ for $s = \frac{2}{3}(\frac{1}{3})^{\frac{2}{3}}$, we calculate $\zeta(s, \frac{1}{4})$ and $\zeta(s, \frac{3}{4})$, using (14), for $s = -\frac{2}{3}(\frac{1}{3})^{\frac{2}{3}}$ with the exception $s = 0$. For $s = 0$ we use the identity $\zeta(0, a) = \frac{1}{2} - a$. The Gamma function, $\Gamma(s)$, appearing on the right side of equation (10) is calculated by making use of the asymptotic series (see [1] Section 12.33)

$$\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log(2\pi) + \sum_{r=1}^{\infty} \frac{(-1)^{r-1} B_r}{2r(2r-1)s^{2r-1}}$$

where B_r are the Bernoulli numbers. This expression holds for $|\arg s| \leq \frac{1}{2}\pi - \Delta$ and $0 < \Delta < \frac{1}{4}\pi$. Using this relation in conjunction with the equation $\Gamma(s+1) = s\Gamma(s)$, we obtain the desired values of the Gamma function.

Equation (17), for $a = 1$, is equivalent to the expression used by Gram [2] and by Haselgrove [3] in preparing their tables of the ordinary Riemann Zeta function.

4. Methods of Checking the Table Values. Equation (16) was used to check the table entries for negative integer values of s , while Hurwitz's formula (9) was used to check the entries for $s = -\frac{2}{3}(\frac{1}{3}) - \frac{2}{3}$. For positive integer values of s , the values of $\zeta(s)$ agree exactly with the table in [5], page xxv. Our table was not checked for other values of s , since no other simple method seemed to present itself. Note, however, the table for negative values of s was calculated using the entries for positive values of s . Hence, it seems safe to assume that the table entries for positive values of s are correct to at least as many digits as the table entries for corresponding negative values of s .

5. Comments Regarding the Calculation of the Table. The table shown below was calculated using a precision of seventy binary digits, which is approximately equivalent to twenty-one decimal digits.

It was impossible to enter most of the values of s exactly in the computer, since $1/3$ does not have a finite binary representation. Therefore, we should consider the errors introduced in our table from using a truncated binary representation. We were able to show, by using a truncated Taylor series, that the errors so introduced did not affect the accuracy of the table. The various calculations involved in showing the insignificance of these errors were not included in this paper, since they were quite straightforward.

6. The Table of Values. The tabular data have been listed to seventeen significant decimal digits. The decimal point is located immediately to the left of the left-most digit of each entry, while the two-digit integer at the right denotes the exponent of the power of ten by which the entry is to be multiplied.

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TABLE 1

s	$\zeta(s, \frac{1}{4})$					$\zeta(s, \frac{3}{4})$				
64/3	69814	63658	33156	77	13	46276	41642	09957	54	03
21	43980	46511	10400	92	13	42044	91481	42985	79	03
62/3	27705	95688	87832	77	13	38200	34049	99724	72	03
61/3	17453	65914	58290	05	13	34707	31296	92456	16	03
20	10995	11627	77601	15	13	31533	68689	82531	57	03
59/3	69264	89222	19591	87	12	28650	25632	40314	70	03
58/3	43634	14786	45735	82	12	26030	48587	06422	96	03
19	27487	79069	44014	41	12	23650	26655	19854	06	03
56/3	17316	22305	54910	39	12	21487	69390	40321	69	03
55/3	10908	53696	61447	33	12	19522	86640	47054	77	03
18	68719	47673	60180	15	11	17737	70232	63351	39	03
53/3	43290	55763	87431	22	11	16115	77333	52502	14	03
52/3	27271	34241	53785	57	11	14642	15330	71921	73	03
17	17179	86918	40225	19	11	13303	28096	72635	09	03
50/3	10822	63940	97051	87	11	12086	83509	03545	36	03
49/3	68178	35603	86554	48	10	10981	62111	35873	48	03
16	42949	67296	02814	98	10	99774	68117	34460	20	02
47/3	27056	59852	45055	63	10	90651	35226	87845	72	02
46/3	17044	58900	99251	96	10	82362	26573	20047	47	02
15	10737	41824	03518	96	10	74831	14030	73469	44	02
44/3	67641	49631	42966	14	09	67988	67020	79914	81	02
43/3	42611	47252	80800	17	09	61771	88735	02511	35	02
14	26843	54560	43992	27	09	56123	58191	92793	72	02
41/3	16910	37408	23656	51	09	50991	77593	55328	03	02
40/3	10652	86813	61045	94	09	46329	24497	93384	86	02
13	67108	86405	50022	12	08	42093	08367	45511	12	02
38/3	42275	93525	33217	14	08	38244	31093	55480	72	02
37/3	26632	17039	13370	57	08	34747	51134	84446	15	02
12	16777	21606	87796	31	08	31570	50939	08321	59	02
35/3	10568	98387	26213	03	08	28684	07349	76745	21	02
34/3	66580	42661	71997	90	07	26061	64725	71060	80	02
11	41943	04086	03548	00	07	23679	10527	14066	46	02
32/3	26422	46042	34022	87	07	21514	53144	78372	37	02
31/3	16645	10745	38096	69	07	19548	01769	36192	33	02
10	10485	76107	68311	48	07	17761	48118	13370	04	02
29/3	66056	16034	78926	59	06	16138	49852	76111	40	02
28/3	41612	77864	98520	10	06	14664	15539	12355	32	02
9	26214	41349	21724	08	06	13324	91014	94593	48	02
26/3	16514	05173	61085	33	06	12108	47045	54080	94	02
25/3	10403	20723	10878	44	06	11003	68161	90407	28	02
8	65536	16938	67090	17	05	10000	42589	27890	58	02
23/3	41285	27582	34437	32	05	90895	31887	04304	92	01
22/3	26008	17634	03920	72	05	82626	93498	64416	39	01
7	16384	21345	51995	72	05	75123	97920	96579	80	01
20/3	10321	50411	38500	09	05	68318	62538	56294	03	01
19/3	65022	44664	70976	70	04	62149	80828	62956	03	01
6	40962	70948	06401	08	04	56562	77857	28816	43	01
17/3	25806	12469	70992	37	04	51508	76680	73079	38	01
16/3	16258	18668	37913	66	04	46944	78141	49920	25	01
5	10243	48974	52658	06	04	42833	58575	64239	47	01
14/3	64546	14107	47051	67	03	39143	93590	59400	43	01
13/3	40679	42509	60218	28	03	35851	22960	35115	28	01

TABLE 1—Continued

s	$\zeta(s, \frac{1}{4})$					$\zeta(s, \frac{3}{4})$				
4	25646	36906	68198	07	03	32938	85422	47510	00	01
11/3	16178	62720	13961	71	03	30400	81320	64265	08	01
10/3	10217	48139	83634	30	03	28246	87653	98801	74	01
3	64663	86996	87684	60	02	26513	16608	16881	98	01
8/3	41092	64168	75447	99	02	25285	74853	60394	04	01
7/3	26334	77863	02380	64	02	24759	96055	02275	82	01
2	17197	32915	45071	11	02	25418	79647	67160	65	01
5/3	11787	30284	47967	72	02	28747	02273	10830	06	01
4/3	95676	33344	56813	25	01	42231	01605	70504	20	01
1		∞					∞			
2/3	-24411	86144	96889	78	00	-20381	06131	82465	56	01
1/3	33101	39009	27282	69	00	-64976	99170	75290	43	00
0	25000	00000	00000	00	00	-25000	00000	00000	00	00
-1/3	13499	19957	79665	53	00	-89579	84483	88113	98	-01
-2/3	55399	96817	77458	36	-01	-19221	97815	54288	52	-01
-1	10416	66666	66666	67	-01	10416	66666	66666	67	-01
-4/3	-10558	77278	36124	00	-01	20147	85800	02319	75	-01
-5/3	-16992	51822	29812	52	-01	20093	86425	54835	96	-01
-2	-15625	00000	00000	00	-01	15625	00000	00000	00	-01
-7/3	-10834	45058	95508	42	-01	97268	16010	89067	10	-02
-8/3	-52815	71878	70004	44	-02	40704	00253	32719	07	-02
-3	-45572	91666	66666	67	-03	-45572	91666	66666	67	-03
-10/3	29301	12394	80509	84	-02	-34597	28641	05474	40	-02
-11/3	46698	19500	95810	35	-02	-48805	58903	88315	47	-02
-4	48828	12500	00000	00	-02	-48828	12500	00000	00	-02
-13/3	38899	36684	63823	16	-02	-37822	44498	22381	09	-02
-14/3	21239	95040	97140	41	-02	-19861	15110	11172	00	-02
-5	60066	34424	60317	46	-04	60066	34424	60317	46	-04
-16/3	-18432	75029	83014	06	-02	19233	18930	79122	45	-02
-17/3	-31960	86810	98371	30	-02	32323	03687	52690	05	-02
-6	-37231	44531	25000	00	-02	37231	44531	24500	00	-02
-19/3	-33017	11705	70240	87	-02	32782	87245	91040	61	-02
-20/3	-19865	27145	97911	51	-02	19530	95592	98213	26	-02
-7	-16148	88509	11458	33	-04	-16148	88509	11458	33	-04
-22/3	22055	37817	57697	42	-02	-22292	83934	16781	00	-02
-23/3	41577	48345	17227	95	-02	-41695	51768	48320	96	-02
-8	52833	55712	89062	50	-02	-52833	55712	89062	50	-02
-25/3	50987	50282	80912	53	-02	-50896	45120	98702	91	-02
-26/3	33229	98874	70615	11	-02	-33088	82387	94668	38	-02
-9	73837	51146	72111	74	-05	73837	51146	72111	74	-05
-28/3	-43631	40309	49183	10	-02	43748	62335	98758	29	-02
-29/3	-88331	79840	49455	97	-02	88394	53105	86276	23	-02
-10	-12045	14503	47900	39	-01	12045	14503	47900	39	-01
-31/3	-12448	12326	45933	15	-01	12442	55605	16962	73	-01
-32/3	-86654	89190	70478	86	-02	86562	63379	78600	91	-02
-11	-51470	93963	85593	44	-05	-51470	93963	85593	44	-05
-34/3	-12956	24917	98495	48	-01	-12964	94737	89903	23	-01
-35/3	27851	69352	45565	33	-01	-27856	63944	24212	63	-01
-12	40274	33693	40896	61	-01	-40274	33693	40896	61	-01
-37/3	44066	74225	89134	15	-01	-44061	81300	66818	43	-01
-38/3	32424	26287	38385	85	-01	-32415	62726	12390	43	-01
-13	50856	42139	11692	30	-05	50856	42139	11692	30	-05
-40/3	-53981	95160	35609	58	-01	53991	01083	71719	25	-01

TABLE 1—Continued

s	$\zeta(s, \frac{1}{4})$					$\zeta(s, \frac{3}{4})$				
-41/3	-12213	66720	37044	43	00	12214	20945	65951	94	00
-14	-18566	93821	02787	49	00	18566	93821	02787	49	00
-43/3	-21331	38708	22892	45	00	21330	79048	86111	17	00
-44/3	-16461	18663	02940	22	00	16460	09042	37713	87	00
-15	-67634	01438	99342	01	-05	-67634	01438	99342	01	-05
-46/3	30053	84266	68572	82	00	-30055	10353	95330	80	00
-47/3	71088	87471	07870	00	00	-71089	66375	73979	33	00
-16	11287	34563	53554	50	01	-11287	34563	53554	50	01
-49/3	13532	13201	24995	85	01	-13532	03739	35347	35	01
-50/3	10887	19301	23970	26	01	-10887	01175	14453	29	01
-17	11649	82235	12560	03	-04	11649	82235	12560	03	-04
-52/3	-21552	81617	38443	47	01	21553	04222	79286	09	01
-53/3	-53020	89805	52583	77	01	53021	04518	12564	35	01
-18	-87489	01292	85995	50	01	87489	01292	85995	50	01
-55/3	-10892	53279	53442	03	02	10892	51375	45982	56	02
-56/3	-90944	16979	46386	51	01	90943	79125	83854	77	01
-19	-25230	56188	58262	09	-04	-25230	56188	58262	09	-04
-58/3	19350	58818	71838	18	02	-19350	63892	61704	53	02
-59/3	49304	35983	54058	91	02	-49304	39403	87055	58	02
-20	84212	65834	78432	17	02	-84212	65834	78432	17	02
-61/3	10846	36397	60161	63	03	-10846	35923	59976	00	03

TABLE 2

s	$\zeta(s)$	s	$\zeta(s)$
64/3	10000 00378 53233 25 01	31/3	10007 87512 59735 22 01
21	10000 00476 93298 68 01	10	10009 94575 12781 81 01
62/3	10000 00600 91541 53 01	29/3	10012 56544 89496 33 01
61/3	10000 00757 13123 99 01	28/3	10015 88198 62306 52 01
20	10000 00953 96203 39 01	9	10020 08392 82608 22 01
59/3	10000 01201 96944 13 01	26/3	10025 41240 15797 39 01
58/3	10000 01514 46249 87 01	25/3	10032 17649 68406 42 01
19	10000 01908 21271 66 01	8	10040 77356 19794 43 01
56/3	10000 02404 35545 91 01	23/3	10051 71613 23892 26 01
55/3	10000 03029 52623 90 01	22/3	10065 66797 08110 32 01
18	10000 03817 29326 50 01	7	10083 49277 38192 28 01
53/3	10000 04809 96364 94 01	20/3	10106 32075 32048 38 01
52/3	10000 06060 86098 24 01	19/3	10135 64087 21870 02 01
17	10000 07637 19763 79 01	6	10173 43061 98444 91 01
50/3	10000 09623 69761 94 01	17/3	10222 34193 45949 71 01
49/3	10000 12127 16680 42 01	16/3	10285 97325 52868 25 01
16	10000 15282 25940 87 01	5	10369 27755 14336 99 01
47/3	10000 19258 75533 22 01	14/3	10479 19229 28351 53 01
46/3	10000 24270 74650 59 01	13/3	10625 64589 86109 19 01
15	10000 30588 23630 70 01	4	10823 23233 71113 82 01
44/3	10000 38551 79067 62 01	11/3	11094 13692 59748 37 01
43/3	10000 48591 05077 17 01	10/3	11473 56236 88274 29 01
14	10000 61248 13505 87 01	3	12020 56903 15959 43 01
41/3	10000 77207 23697 67 01	8/3	12841 90540 23974 21 01
40/3	10000 97332 08000 25 01	7/3	14151 55609 44598 30 01
13	10001 22713 34757 85 01	2	16449 34066 84822 64 01
38/3	10001 54728 79053 14 01	5/3	21235 22968 85758 35 01
37/3	10001 95119 46821 60 01	4/3	36009 37750 45886 24 01
12	10002 46086 55330 80 01	1	∞
35/3	10003 10414 39287 00 01	2/3	-24475 80736 23365 82 01
34/3	10003 91627 16273 87 01	1/3	-97336 02483 50782 72 01
11	10004 94188 60411 95 01	0	-50000 00000 00000 00 00
32/3	10006 23757 16251 28 01		