

Note on Latent Roots and Vectors of Segments of the Hilbert Matrix

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As a by-product of work on condition numbers, maximum and minimum latent roots, and the corresponding vectors, have been calculated for the Hilbert segments of orders 4, 6, 8, and 10, to about 17 significant digits.

This supplements the data of Fairthorne and Miller [1] by including minimum latent roots and corresponding vectors, and increasing the accuracy. It also verifies their data (after rounding) except that one error was found (see next paragraph) in their published results.

A Hilbert segment of order N is the matrix $\| 1/(m+n-1) \|; m, n = 1, \dots, N$. Let H represent a Hilbert segment, T its inverse, λ a latent root, and V a corresponding vector. Then, e.g., $\lambda_1(H_6)$ is the largest latent root of the segment of order 6; $V_N(T_6)$ is the vector corresponding to the smallest latent root of the inverse of H_6 . The error in Fairthorne and Miller's article occurs in the third element of $V_1(H_6)$.

The power method [2] was used, with a double-precision floating-point routine on the IBM 650. Smallest latent roots were obtained as $\lambda_N(H_N) = 1/\lambda_1(T_N)$, and verified (to as many significant figures as the method allows) by direct calculation of $\lambda_N(H_N)$ by the power method, $\lambda_N(H)$ being obtained as $\lambda_1(H - pI)$, where I is the identity matrix and p is slightly greater than $\lambda_1(H)$. The N th vectors are given as $V_N(H_N) = V_1(T_N)$, because the method gives greater accuracy here for V_1 than for V_N and because T has no input error.

Terminal digits are uncertain by not more than one, as indicated by convergence rates.

	$\lambda_1(H_4)$		$\lambda_N(H_4)$
1. 50021 42800 59242 81		$10^{-4} \times$	0. 96702 30402 25868 861
	$V_1(H_4)$		$V_N(H_4)$
1.			0. 03688 76826 14141 047
0. 57017 20836 63235 83			-0. 41534 92877 80311 17
0. 40677 89880 27529 24			1
0. 31814 09688 73793 96			-0. 65017 12197 33679 82
	$\lambda_1(H_6)$		$\lambda_N(H_6)$
1. 61889 98589 24339 1		$10^{-7} \times$	1. 08279 94845 65549 8
	$V_1(H_6)$		$V_N(H_6)$
1			0. 00180 94825 41440 515
0. 58862 85434 25543 2			-0. 05161 82535 94248 58
0. 42832 72844 28956 1			0. 34890 77525 35503 9
0. 33966 18918 38709 5			-0. 90671 76845 78412 7
0. 28252 35879 42149 2			1
0. 24233 78111 22849 5			-0. 39374 11114 93702 0
	$\lambda_1(H_8)$		$\lambda_N(H_8)$
1. 69593 89969 21949 4		$10^{-10} \times$	1. 11153 89663 72442 4

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$V_1(H_8)$			$V_N(H_8)$			
1.			-0.00006	86103	92145	12812
0.60050	42457	57953	0.00368	78770	51827	661
0.44267	15540	11918	-0.04826	72545	24498	43
0.35437	04469	99697	0.26171	33996	76104	1
0.29691	85784	44507	-0.70574	73471	79618	8
0.25618	09294	86980	1.			
0.22562	93688	08227	-0.71250	91381	80124	8
0.20179	01870	37918	0.20124	18343	83776	4
	$\lambda_1(H_{10})$			$\lambda_N(H_{10})$		
1.75191	96702	65177	$10^{-18} \times$	1.09315	38198	57659
	$V_1(H_{10})$			$V_N(H_{10})$		
1.			0.00000	27147	13133	60409
0.60899	19143	69650	-0.00023	60612	62959	0383
0.45313	82989	59421	0.00505	28973	86716	890
0.36528	60134	02151	-0.04611	60400	49989	25
0.30775	30474	45501	0.22066	15177	28910	4
0.26672	51842	93050	-0.60817	67839	54336	8
0.23580	13079	82484	1.			
0.21156	39639	51540	-0.96815	88795	12219	1
0.19200	51281	86119	0.50907	35851	67138	3
0.17586	00343	93102	-0.11210	49402	14747	4

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1. R. A. FAIRTHORNE & J. C. P. MILLER, "Hilbert's double series theorem and principal latent roots of the resulting matrix," *MTAC*, v. 3, 1949, p. 399.
 2. MARVIN MARCUS, "Basic theorems in matrix theory," *Nat. Bur. Standards, Appl. Math. Ser. No. 57*, U. S. Government Printing Office, Washington, D. C., 1960.