Euler’s Constant to 1271 Places

By Donald E. Knuth

Abstract. The value of Euler’s or Mascheroni’s constant
\[ \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \right) \]
has now been determined to 1271 decimal places, thus extending the previously
known value of 328 places. A calculation of partial quotients and best rational
approximations to \( \gamma \) was also made.

1. Historical Background. Euler’s constant was, naturally enough, first evaluated
by Leonhard Euler, and he obtained the value 0.577218 in 1735 [1]. By 1781 he had
calculated it more accurately as 0.5772156649015325 [2]. The calculations were
carried out more precisely by several later mathematicians, among them Gauss,
who obtained
\[ \gamma = 0.57721566490153286060653. \]

Various British mathematicians continued the effort [3], [4]; an excellent account
of the work done on evaluation of \( \gamma \) before 1870 is given by Glaisher [5]. Finally,
the famous mathematician-astronomer J. C. Adams [6] laboriously determined
\( \gamma \) to 263 places. Adams thereby extended the work of Shanks, who had obtained
110 places (101 of which were correct).

Although much work has been done trying to decide whether \( \gamma \) is rational,
the evaluation has not been carried out any more precisely. With the use of high-speed
computers, the constants \( \pi \) and \( e \) have been evaluated to many thousands of decimal
places [11], [12]. A complete bibliography for \( \pi \) appears in [11]. The evaluation of
\( \gamma \) to many places is considerably more difficult.

2. Evaluation of \( \gamma \). The technique used here to calculate \( \gamma \) is essentially that used
by Adams and earlier mathematicians. A complete derivation of the method is
given by Knopp [7]. We use Euler’s summation formula in the form
\[
\sum_{i=1}^{n} f(i) = \int_{1}^{n} f(x) \, dx + \frac{1}{2} (f(n) + f(1)) + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} \left[ f^{(2j-1)}(n) - f^{(2j-1)}(1) \right] + R_k
\]
where \( B_m \) are the Bernoulli numbers defined symbolically by
\[ e^{Bx} = \frac{x}{e^x - 1}. \]

With this notation, \( B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \) etc. Here the remainder

Received January 12, 1962.
\[ R_k \text{ is given by} \]
\[
(3) \quad R_k = \frac{1}{(2k+1)!} \int_1^n P_{2k+1}(x) f^{(2k+1)}(x) \, dx;
\]
and \( P_{2k+1}(x) \) is a periodic Bernoulli polynomial, symbolically
\[
(4) \quad P_{2k+1}(x) = ([x] + B)^{2k+1} = (-1)^{k-1}(2k+1)! \sum_{\ell=1}^{\infty} \frac{2 \sin 2\pi \ell x}{(2\pi)^{2k+1}}
\]
where \([x]\) is the fractional part of \( x \).

Now we put \( f(x) = 1/x \), obtaining from (1)
\[
1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \frac{1}{2} + \frac{1}{2n} \]
\[
+ \frac{B_2}{2} \left(1 - \frac{1}{n^2}\right) + \cdots + \frac{B_{2k}}{2k} \left(1 - \frac{1}{n^{2k}}\right) - \int_1^n P_{2k+1}(x) \, dx.
\]
Taking the limit in (5) as \( n \to \infty \), we find
\[
(6) \quad \gamma = \frac{1}{2} + \frac{B_2}{2} + \cdots + \frac{B_{2k}}{2k} - \int_1^\infty \frac{P_{2k+1}(x)}{x^{2k+2}} \, dx.
\]
Subtracting (5) from (6) gives
\[
(7) \quad \gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \frac{1}{2n} + \frac{B_2}{2n^2} \]
\[
+ \cdots + \frac{B_{2k}}{(2k)n^{2k}} - \int_1^n \frac{P_{2k+1}(x)}{x^{2k+2}} \, dx.
\]
If the remainder is discarded and we consider (7) as an infinite series in \( k \), it diverges as \( k \to \infty \). It still yields a good method for calculating \( \gamma \), however, since
\[
(8) \quad | P_{2k+1}(x) | \leq \frac{2(2k+1)!}{(2\pi)^{2k+1}} \sum_{\ell=1}^{\infty} \frac{1}{\ell^{2k+1}},
\]
and by applying Stirling's formula to (8) we obtain
\[
(9) \quad \left| \int_n^\infty \frac{P_{2k+1}(x)}{x^{2k+2}} \, dx \right| \leq \frac{4}{n} \sqrt{k} \left( \frac{k}{n\pi e} \right)^{2k}.
\]
Put \( k = 250 \) and \( n = 10000 \) to obtain a remainder
\[
(10) \quad \left| \int_{10000}^\infty \frac{P_{999}}{x^{502}} \, dx \right| < 10^{-1269},
\]
so these values may be used in (7) to determine \( \gamma \) to at least 1269 places. This particular choice of \( k \) and \( n \) was made for convenience on a decimal computer, in an attempt to obtain the greatest precision in a reasonable time.

3. Details of the Computation. The sum \( 1 + \frac{1}{2} + \cdots + \frac{1}{10000} \) was evaluated as
\[
(11) \quad S_{10000} = \frac{3}{2} + \frac{7}{12} + \cdots + \frac{199999}{99990000} = 9.787606036 \cdots.
\]
Combining terms in this way reduced the number of necessary divisions. The natural
logarithm of 10000 was then determined by
\[(12) \quad \ln 10000 = -252 \ln (1 - .028) + 200 \ln (1 + .0125) + 92 \ln (1 - .004672).\]
Such an expansion was designed for fast convergence and for convenience on a
decimal computer. It is a simple matter to obtain such an expansion by hand
calculation; we seek integers \((x, y, z)\) such that \(2^x3^y5^z \approx 1\) and \(y \geq 0\). If three linearly
independent solutions are obtained, one can calculate \(\ln 2\), \(\ln 3\), and \(\ln 5\), and, in
particular, \(\ln 10\). If \(2^x3^y5^z > 1\) and \(2^x3^y5^z < 1\), suitable positive integral com-
binations of \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) will give closer approximations. The
method is to find small values of \((x, y, z)\) so that \(x + y \log_2 3 + z \log_2 5 \approx 0\), then
combine these to get better and better approximations. The expansion (12) cor-
responds to the solutions \((-1, 5, -3), (-4, 4, -1),\) and \((6, 5, -6)\). For a binary
computer the extra requirement \(z \geq 0\) makes it more difficult, but solutions can be
used as such
\[(12a) \quad \ln 10000 = 160 \ln 2^{-32}3^{75} - 864 \ln 2^{-11}3^{52} + 292 \ln 2^{-15}3^{5}.\]
Finally, Bernoulli numbers \(B_{2k} = 10^{-sk}B_{2k}\) were evaluated using the recursion
relation
\[(13) \quad \left(\frac{2k + 1}{2k}\right)B'_{2k} + 10^{-8}\left(\frac{2k + 1}{2k - 2}\right)B'_{2k-2} + \cdots + 10^{8-8k}\left(\frac{2k + 1}{2}\right)B_{2k'} = \frac{(2k - 1)}{2 \cdot 10^{sk}}.\]
From the fact that
\[(14) \quad \frac{|B_{2k}|}{B_{2k-2}} \approx \frac{2k(2k - 1)}{4\pi^2}\]
it can be seen that the recursion (13) does not cause truncation errors to propagate.
Furthermore, 1300 decimal places were used in all calculations.
When using (13) to calculate \(B_{2k}\), first all the positive terms were added to-
gether, then all the negative terms added together and finally the two were com-
bined. This gave extra speed to the calculations. Care was also taken to avoid
multiplying by zero. The evaluation of \(B'_{2k}\) becomes more difficult as \(k\) increases,
because of the number of terms and the size of the binomial coefficients. Since the
\(B_{2k}\) alternate in sign, the actual error in the calculation of \(\gamma\) is less than \(\frac{B_{102}}{502} \approx +0.25 \times 10^{-1271}\), so the value obtained here should be correct to 1271 decimals. The fact
that the final answer agreed with Adams' value and that numerous checks were
made on all the arithmetical routines provides a good basis for guaranteeing the
stated accuracy of the results. Dr. Wrench has independently verified the approxi-
mations to 1039 decimal places.
The present calculations were performed on a Burroughs 220 computer. The
evaluation of \(S_{10000}\) required approximately one hour, and each of the logarithms
required about six minutes. Evaluation of the 250 Bernoulli numbers was the
most troublesome part of the calculations, and the total time for their calculation
was approximately eight hours. A table of the Bernoulli numbers \(B'\) to 1270D has
been sent to the Unpublished Mathematical Tables file of the journal, Mathematics
of Computation.
4. Determination of Partial Quotients. To find best rational approximations to \( \gamma \), we represent it as a continued fraction

\[
\gamma = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}.
\]

Put \( P_1 = Q_0 = 1, Q_1 = P_0 = 0 \), and for \( i \geq 1 \)

\[
P_{i+1} = a_i P_i + P_{i-1}, \quad Q_{i+1} = a_i Q_i + Q_{i-1}.
\]

In matrix notation,

\[
\begin{pmatrix}
P_{i+1} \\
Q_{i+1}
\end{pmatrix} =
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}^i
\begin{pmatrix}
P_1 \\
Q_1
\end{pmatrix} =
\begin{pmatrix}
a_1 & 1 \\
1 & 0
\end{pmatrix} \cdots \begin{pmatrix}
a_i & 1 \\
1 & 0
\end{pmatrix}.
\]

Then \( \lim P_i/Q_i = \gamma \). The fractions \( P_i/Q_i \) represent the best approximations to \( \gamma \) in the sense that

\[
| Q_i \gamma - P_i | < | q \gamma - p |, \quad \text{if} \quad q < Q_i, \quad i \geq 3.
\]

We have then \( a_i \neq 0 \) \( (i > 1) \), if and only if \( \gamma \) is irrational, and the sequence of partial quotients \( a_i \) will be periodic if and only if \( \gamma \) is quadratic, that is,

\[
\gamma = r + \sqrt{s}, \quad r \text{ and } s \text{ rational.}
\]

For proofs of these well-known results see Cassels [8].

The algorithm used to determine the partial quotients \( a_i \), using limited decimal precision, is as follows:

Set \( \gamma_1 = \gamma \), and

\[
a_i = [\gamma_i], \quad \gamma_{i+1} = [\gamma_i]^{-1}, \quad i \geq 1.
\]

We have decimal numbers \( r_i \) and \( s_i \) such that

\[
r_i \leq \gamma_i \leq s_i.
\]

We successively find numbers \( r_i, s_i \) such that

\[
r_i \leq \gamma_i \leq s_i.
\]

If \( |r_i| \neq |s_i| \), then the algorithm terminates. If \( |r_i| = |s_i| \), then \( |r_i| = a_i \), and

\[
|r_i| \leq |\gamma_i| \leq |s_i|.
\]

Hence

\[
|s_i|^{-1} \leq \gamma_{i+1} \leq |r_i|^{-1}.
\]

Choose decimal numbers \( r_{i+1} \) and \( s_{i+1} \) so that \( r_{i+1} \leq |s_i|^{-1} \) by truncation, \( s_{i+1} \geq |r_i|^{-1} \) by rounding up. Then the algorithm continues, until \( |r_i| \neq |s_i| \).

The method used for calculating \( |s_i|^{-1} \) when \( |s_i| \) has several hundred decimal places was adapted from that of Pope and Stein [10]. Approximately six seconds was required to obtain each quotient. If \( t \) partial quotients are desired, the total time is proportional to \( t^2 \).
Table 1 gives the value of $\gamma$ to 1271 decimal places. Table 2 gives the first 372 partial quotients of $\gamma$. Only 372 are given, although the value in Table 1 would have probably yielded over 1000 partial quotients. Table 3 gives for the reader’s convenience the first few “best rational approximations” to $\gamma$. Here the ratio 228/395 gives a remarkably good value, correct to six decimal places.

From Table 2 one can compute

$Q_{373} \approx 1.135 \times 10^{199},$

and we can conclude that if $\gamma$ is rational its denominator must be larger than $Q_{373}$. Another consequence is that only about 385 decimal places of Table 1 were needed to obtain the 372 partial quotients. The referee has pointed out that Lehman [14] had already calculated the first 315 partial quotients for $\gamma$ on the basis of Wrench’s 328-place value [9]. These are in perfect agreement with the values obtained here.

The partial quotients of $\gamma$, as calculated in Table 2, appear to be “random” in some sense. Almost all real numbers have partial quotients satisfying

$$\lim \sqrt[n]{a_2 a_3 \cdots a_{n+1}} = K$$

where $K \approx 2.685$ is Khintchine’s constant [13]. In this case,

$$\sqrt[371]{a_2 a_3 \cdots a_{372}} \approx 2.692,$$

a reasonable approximation to $K$.

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Table 3

Best Rational Approximations

| 1/2 | .50 |
| 3/5 | .60 |
| 4/7 | .571 |
| 11/19 | .579 |
| 15/26 | .5769 |
| 71/123 | .57724 |
| 228/395 | .5772152 |
| 3035/5258 | .57721567 |
| 15403/26685 | .5772156642 |
| 18438/31943 | .5772156654 |
| 33841/58628 | .57721566487 |

The author wishes to acknowledge his gratitude to the Burroughs Corporation and to the Case Institute of Technology for the use of their Burroughs 220 computers.

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