Expansions of Jacobian Elliptic Functions in Powers of the Modulus

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The Maclaurin expansions for Jacobian elliptic functions in the neighborhood of zero modulus (to be called the k-expansions) are given herein. Since the absolute value of modulus k can be taken as smaller than one without loss of generality, the k expansions in most cases converge faster than similar expansions in the neighborhood of the zero argument [1] (to be called the u-expansions). The relative accuracies of the two types of expansions will be illustrated numerically for several pairs of k and u values and by taking equal numbers of terms in the computation. The k expansions also provide for the information, in a fundamental way, as to how the Jacobian functions vary with the change of modulus. The information may be useful in the study of damped non-linear vibrations.

The basic Jacobian function $sn(u, k^2)$ is related to its argument, u, and modulus, k, as the following:

$$u = \int_0^{sn(u,k^2)} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = \int_0^{sn(u,\phi)} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\phi t^2}}. \tag{1}$$

By successive differentiations one obtains

$$\left\{\begin{aligned}
\frac{\partial sn}{\partial \phi} &= -cn\cdot dn \int_0^{sn} \frac{(t^2/2)\,dt}{\sqrt{(1-t^2)(1-\phi t^2)^3}}\\
\frac{\partial^2 sn}{\partial \phi^2} &= -cn\cdot dn \left[\int_0^{sn} \frac{1\cdot 3(t^2/2)^2\,dt}{\sqrt{(1-t^2)(1-\phi t^2)^5}}\right.\\
&\qquad\qquad \left. + \frac{sn^2}{2cn\cdot dn^3}\cdot\frac{\partial sn}{\partial\phi} + \frac{\partial sn}{\partial\phi}\cdot\frac{\partial}{\partial\phi}\left(\frac{1}{cn\cdot dn}\right)\right]\\
&\cdots\\
\frac{\partial^m sn}{\partial \phi^m} &= -cn\cdot dn \left\{\int_0^{sn} \frac{1\cdot 3\cdot 5\cdots(2m-1)(t^2/2)^m}{\sqrt{(1-t^2)(1-\phi t^2)^{2m+1}}}\,dt\right.\\
&\quad + \frac{\partial sn}{\partial\phi}\left[\frac{1\cdot 3\cdot 5\cdots(2m-3)(sn^2/2)^{m-1}}{cn\cdot dn^{2m-1}}\right.\\
&\quad \left. + \frac{\partial}{\partial\phi}\left(\frac{1\cdot 3\cdot 5\cdots(2m-5)(sn^2/2)^{m-2}}{cn\cdot dn^{2m-3}}\right) + \cdots + \frac{\partial^{m-1}}{\partial\phi^{m-1}}\left(\frac{1}{cn\cdot dn}\right)\right]\\
&\quad + \frac{\partial^2 sn}{\partial\phi^2}\left[\frac{1\cdot 3\cdot 5\cdots(2m-5)(sn^2/2)^{m-2}}{cn\cdot dn^{2m-3}}\right.\\
&\quad + 2\frac{\partial}{\partial\phi}\left(\frac{1\cdot 3\cdot 5\cdots(2m-7)(sn^2/2)^{m-3}}{cn\cdot dn^{2m-5}}\right)\\
&\quad \left. + \cdots + (m-1)\frac{\partial^{m-2}}{\partial\phi^{m-2}}\left(\frac{1}{cn\cdot dn}\right)\right] + \cdots\\
&\quad \left. + \frac{\partial^{m-1} sn}{\partial\phi^{m-1}}\left[(m-1)\frac{\partial}{\partial\phi}\left(\frac{1}{cn\cdot dn}\right)\right]\right\}.
\end{aligned}\right. \tag{2}$$

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It follows upon evaluating these derivatives at $\phi = 0$ and noting that $\phi = k^2$,

$$
\begin{aligned}
sn(u, k^2) &= \sin u - \frac{\cos u}{4}\left(u - \frac{\sin 2u}{2}\right)k^2 \\
&\quad - \frac{\cos u}{4}\left[\frac{3}{4}\left(\frac{3u}{2} - \sin 2u + \frac{\sin 4u}{8}\right) - \sin^2 u\left(u - \frac{\sin 2u}{2}\right)\right. \\
&\quad \left. + \frac{\tan u}{4}\left(u - \frac{\sin 2u}{2}\right)^2\right]\frac{k^4}{2!} - \cdots + \left\{\frac{\partial^m sn}{\partial \phi^m}\right\}_{\phi=0} \cdot \frac{k^{2m}}{m!} - \cdots.
\end{aligned} \tag{3}
$$

Expansions for other Jacobian functions can be generated from the relations between these functions and sn. The results are given explicitly up to k^4 in the following:

$$
\begin{aligned}
cn(u, k^2) &= \cos u + \frac{\sin u}{4}\left(u - \frac{\sin 2u}{2}\right)k^2 - \left\{\frac{\cos u}{16}\left(u - \frac{\sin 2u}{2}\right)^2\right. \\
&\quad \left. + \frac{\sin u}{4}\left[\sin^2 u\left(u - \frac{\sin 2u}{2}\right) - \frac{3}{4}\left(\frac{3}{2}u - \sin 2u + \frac{\sin 4u}{8}\right)\right]\right\}\frac{k^4}{2!} + \cdots.
\end{aligned} \tag{4}
$$

$$
dn(u, k^2) = 1 - \frac{1}{2}(\sin^2 u)k^2 + \left(\frac{u \sin 2u}{4} - \frac{\sin^2 2u}{8} - \frac{1}{4}\sin^4 u\right)\frac{k^4}{2!} + \cdots. \tag{5}
$$

$$
\begin{aligned}
ns(u, k^2) &= \csc u + \csc^2 u\left(\frac{\cos u}{4}\right)\left(u - \frac{\sin 2u}{2}\right)k^2 + \left[-\frac{\cos u}{4}\left(u - \frac{\sin 2u}{2}\right)\right. \\
&\quad - \frac{3}{16}\cos u \csc^2 u\left(\frac{3}{2}u - \sin 2u + \frac{\sin 4u}{8}\right) \\
&\quad \left. + \frac{1}{16}\csc u(\csc^2 u + \cot^2 u)\left(u - \frac{\sin 2u}{2}\right)^2\right]\frac{k^4}{2!} + \cdots.
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
cs(u, k^2) &= \cot u + \frac{\csc^2 u}{4}\left(u - \frac{\sin 2u}{2}\right)k^2 + \left[-\frac{1}{8}\left(u - \frac{\sin 2u}{2}\right)^2 \cot u \csc^2 u\right. \\
&\quad \left. + \frac{3}{16}\csc^2 u\left(\frac{3}{2}u - \sin 2u + \frac{\sin 4u}{8}\right) - \frac{1}{4}\left(u - \frac{\sin 2u}{2}\right)\right]\frac{k^4}{2!} + \cdots.
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
ds(u, k^2) &= \csc u + \left[\frac{1}{4}\cot u \csc u\left(u - \frac{\sin 2u}{2}\right) - \frac{1}{2}\sin u\right]k^2 \\
&\quad + \frac{1}{4}\left[-\sin^3 u + \left(u - \frac{\sin u}{2}\right)^2\left(\frac{\csc u}{4}\right)(\csc^2 u + \cot^2 u)\right. \\
&\quad \left. + \frac{3}{4}\cot u \csc u\left(\frac{3}{2}u - \sin 2u + \frac{\sin 4u}{8}\right)\right]\frac{k^4}{2!} + \cdots.
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
sc(u, k^2) &= \tan u - \frac{1}{4}\sec^2 u\left(u - \frac{\sin 2u}{2}\right)k^2 \\
&\quad - \frac{1}{4}\left[\frac{3}{8}(1 + \sec^2 u)\left(\frac{3}{2}u - \sin 2u + \frac{\sin 4u}{8}\right)\right. \\
&\quad \left. - \left(u - \frac{\sin 2u}{2}\right)\tan^2 u - \frac{1}{2}\left(u - \frac{\sin 2u}{2}\right)^2 \tan u \sec^2 u\right]\frac{k^4}{2!} - \cdots.
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
dc(u, k^2) &= \sec u - \frac{1}{4}\tan u\left[2\sin u + \sec u\left(u - \frac{\sin 2u}{2}\right)\right]k^2 \\
&\quad + \left[\frac{\sec u}{4}\left(u\sin 2u - \frac{\sin^2 2u}{2} - \sin^4 u\right) + \frac{1}{2}\tan^2 u \sin u\left(u - \frac{\sin 2u}{2}\right)\right. \\
&\quad - \frac{3}{16}\sin u \sec^2 u\left(\frac{3}{2}u - \sin 2u + \frac{\sin 4u}{8}\right) \\
&\quad \left. - \frac{1}{16}\sec u\left(u - \frac{\sin 2u}{2}\right)^2(\tan^2 u + \sec^2 u)\right]\frac{k^4}{2!} + \cdots .
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
nc(u, k^2)) &= \sec u - \frac{\tan u \sec u}{4}\left(u - \frac{\sin 2u}{2}\right)k^2 \\
&\quad + \left[\left(\frac{1}{16}\sec u + \frac{1}{8}\tan^2 u \sec u\right)\left(u - \frac{\sin 2u}{2}\right)^2 + \frac{1}{4}\tan^2 u \sin u\left(u - \frac{\sin 2u}{2}\right)\right. \\
&\quad \left. + \frac{3}{4}\sec^2 u\left(\frac{3}{2}u - \sin 2u + \frac{\sin 4u}{8}\right)\right]\frac{k^4}{2!} + \cdots .
\end{aligned} \tag{11}
$$

$$
nd(u, k^2) = 1 + \frac{\sin^2 u}{2}k^2 - \frac{1}{4}\left[u\sin 2u - \frac{\sin^2 2u}{2} - 3\sin^4 u\right]\frac{k^4}{2!} + \cdots . \tag{12}
$$

TABLE 1
Computed Jacobian Elliptic Functions

u	$\mathrm{Sin}^{-1} k$	*sn*					*cn*				
		k-Series		u-Series		Exact Value	k-Series		u-Series		Exact Value
		Computed Value	% Error	Computed Value	% Error		Computed Value	% Error	Computed Value	% Error	
.1745	5°	.17361	.02	.17361	.02	.17365	.98481	0	.98481	0	.98481
.1754	80°	.17365	0	.17365	0	.17365	.98481	0	.98654	.18	.98481
.7016	15°	.64277	0	.64294	.02	.64279	.76606	0	.76668	.08	.76604
.9401	50°	.76597	.01	.77886	1.67	.76604	.64282	0	.66704	3.77	.64279
1.1424	45°	.86587	.02	.90344	4.32	.86603	.50024	.05	.56037	12.07	.50000
1.3372	40°	.93964	.01	1.02184	8.74	.93969	.34250	.14	.45935	34.3	.34202
1.5738	5°	.99999	0	1.000821	.82	1.0000	0	0	.02496		0

u	$\mathrm{Sin}^{-1} k$	*dn*				
		k-Series		u-Series		Exact Value
		Computed Value	% Error	Computed Value	% Error	
.1745	5°	.99986	0	.99988	0	.99988
.1754	80°	.98525	0	.98698	.17	.98527
.7016	15°	.98606	0	.98626	.02	.98606
.9401	50°	.80898	.09	.82828	2.29	.80972
1.1424	45°	.79147	.12	.83341	5.42	.79056
1.3372	40°	.80027	.41	.87352	9.61	.79697
1.5738	5°	.99620	0	.99837	.22	.99619

$$
\begin{aligned}
cd(u, k^2) = \cos u + \frac{\sin u}{4}\left(u + \frac{\sin 2u}{2}\right)k^2 + \Bigg[-\frac{\cos u}{16}\left(u - \frac{\sin 2u}{2}\right)^2 \\
+ \frac{3}{16}\sin u\left(\frac{3}{2}u - \sin 2u + \frac{\sin 4u}{8}\right) \\
- \frac{\cos u}{4}\left(u \sin 2u - \frac{\sin^2 2u}{2} - 3\sin^4 u\right)\Bigg]\frac{k^4}{2!} + \cdots .
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
sd(u, k^2) = \sin u - \left[\frac{\cos u}{4}\left(u - \frac{\sin 2u}{2}\right) - \frac{\sin^3 u}{2}\right]k^2 \\
- \left\{\frac{\cos u}{4}\left[\frac{3}{4}\left(\frac{3u}{2} - \sin 2u + \frac{\sin 4u}{8}\right) + \frac{\tan u}{4}\left(u - \frac{\sin 2u}{2}\right)^2\right]\right. \\
\left. + \frac{\sin u}{4}\left(u \sin 2u - \frac{\sin^2 2u}{2} - 3\sin^4 u\right)\right\}\frac{k^4}{2!} - \cdots .
\end{aligned}
\tag{14}
$$

Representative results computed for basic Jacobian functions using the k-series as well as the u-series are given in Table 1. The computations were carried to the first three terms for both types of series. The various k-u values, representing typical combinations in different ranges, were picked from the table of elliptic integrals [2], so that the "exact" values of the functions are known and that the relative accuracies of the two types of series may be compared. The percentage errors are computed to the nearest .01. It is seen that the k-series are nearly always more accurate than the corresponding u-series, particularly at larger u-values.

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1. H. Bateman, *Higher Transcendental Functions*, v. II., McGraw-Hill Book Company, New York, 1953, p. 344.
2. E. Jahnke & F. E. Emde, *Table of Functions*, Fourth Edition, Dover, New York, 1945.