

Sign Wave Analysis in Matrix Eigenvalue Problems

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1. Introduction. In this paper the word "matrix" denotes a real (but not necessarily symmetric) matrix of order N ; by "vector" we mean a column vector with real or complex elements. For any matrix A , the roots of the equation $\det(A - \lambda I) = 0$ ($I =$ unit matrix) are called the *eigenvalues* of A . By the *multiplicity* of an eigenvalue we mean its multiplicity as a root of the above polynomial equation. If λ is an eigenvalue of A , then any nontrivial solution x of the equation $(A - \lambda I)x = 0$ is called an eigenvector (of A) associated with λ . An eigenvalue is a *dominant eigenvalue* of the matrix A if its modulus is exceeded by the modulus of no other eigenvalue of A .

The power method ([1]; [3], p. 296; [5]; [7]; [9]; [10]) is generally recognized as a numerically efficient algorithm for determining the dominant eigenvalue(s) and associated eigenvector(s) of a matrix. We review the method briefly for the case where the matrix A has a single dominant eigenvalue λ with associated eigenvector u . (It is assumed that λ has multiplicity one.) Denoting by the superscript T the transpose of a vector or matrix, we let v be an eigenvector of A^T associated with the eigenvalue λ . Starting with any vector $x^{(0)}$ satisfying $v^T x^{(0)} \neq 0$, we now form by successive matrix-vector multiplications the vectors

$$x^{(n+1)} = Ax^{(n)}, \quad n = 0, 1, 2, \dots$$

Then, denoting by a_ν ($\nu = 1, 2, \dots, N$) the components of a vector a , we have for every ν such that $u_\nu \neq 0$

$$\lim_{n \rightarrow \infty} \frac{x_\nu^{(n+1)}}{x_\nu^{(n)}} = \lambda,$$

and furthermore, as $n \rightarrow \infty$,

$$x_1^{(n)} : x_2^{(n)} : \dots : x_N^{(n)} \rightarrow u_1 : u_2 : \dots : u_N. \quad *$$

The convergence of the process can be sped up by devices such as shift of the origin [10], fractional iteration [8], and the δ^2 -process [1]. Statements similar to the above still hold if the multiplicity of λ is greater than one, but the convergence may then be slow due to the presence of nonlinear divisors. Once λ , u , and the associated eigenvector of A^T have been determined, one can, by a process known as deflation, construct a matrix A_1 whose eigenvalues and eigenvectors are the same as those of A , except that the eigenvalue λ is replaced by 0. The above process can

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* This notation means that

$$\frac{x_\nu^{(n)}}{\left[\sum_{\mu=1}^N |x_\mu^{(n)}|^2 \right]^{1/2}} \rightarrow \frac{u_\nu}{\left[\sum_{\mu=1}^N |u_\mu|^2 \right]^{1/2}} \quad (n \rightarrow \infty, \nu = 1, 2, \dots, N).$$

then be repeated; if the matrices A_1, A_2, \dots all have single dominant eigenvalues, the above method yields successively all eigenvalues and eigenvectors of the matrix A .

2. Conjugate Complex Dominant Eigenvalues. In the present note we wish to deal with the case where the matrix A has exactly two dominant eigenvalues, both simple, represented by the pair of conjugate complex numbers $\lambda = \rho e^{i\varphi}$ and $\bar{\lambda} = \rho e^{-i\varphi}$, where $\rho > 0, 0 < \varphi < \pi$. The eigenvectors associated with λ and with $\bar{\lambda}$ may then be assumed to be conjugate complex vectors also. We shall denote them by u and \bar{u} , where the components u_ν of u are given by

$$u_\nu = r_\nu e^{i\varphi_\nu}, \quad \nu = 1, 2, \dots, N.$$

In view of the fact that the eigenvectors are determined only up to a non-zero factor (which in the present case may even be complex), it should be noted that the r_ν are determined only up to a positive factor, and the φ_ν only up to a common additive constant modulo 2π . Indicating by the superscript H the conjugate transpose of a complex vector or matrix, we denote by v the eigenvector of A^H belonging to λ , normalized such that $v^H u = 1$.

One of the methods for determining conjugate complex eigenvalues and corresponding eigenvectors from the sequence $\{x^{(n)}\}$ that have been proposed ([3], p. 296; [9]) is known to be numerically unstable for small values of φ [9]. In Section 3 below we propose an alternate method that appears to be uniformly accurate for all values of φ . In addition, the method yields very good approximations for both φ and the φ_ν almost without computation, by mere inspection of the signs of the sequences of the components of the (real) vectors $x^{(n)}$.

3. Sign Waves. It is known ([2], p. 285) that the presence of a pair of conjugate complex dominant eigenvalues is indicated by the occurrence of sign changes in the sequences $\{x_\nu^{(n)}\}$. For a certain matrix of order 6, the signs of the $x_\nu^{(n)}$ were distributed as follows:

ν	1	2	3	4	5	6
$n = 0$	+	+	+	+	+	+
1	+	+	+	-	+	+
2	-	-	-	-	-	+
3	-	-	-	+	-	-
4	+	+	-	+	+	-
5	+	+	+	-	+	+
6	-	-	+	-	-	+
7	-	-	-	+	-	-
8	-	-	-	+	-	-
9	+	+	+	-	+	+
10	+	+	+	-	+	+
\vdots						

Our method consists in exploring systematically the period and relative location of these *sign waves*. We ascribe a sign to *all* elements of the N sequences

$\{x_\nu^{(n)}\}$ ($\nu = 1, \dots, N$). A zero element is assigned the sign of the first nonzero element following it. (If there is no such element, the sign is irrelevant for the following theory.) For $k = 1, 2, \dots$ we denote by $n_\nu^{(k)}$ the index of the element in the sequence $\{x_\nu^{(n)}\}$ at which the sign changes from minus to plus for the k th time, i.e., at which

$$\text{sign } x_\nu^{(n-1)} = -1, \quad \text{sign } x_\nu^{(n)} = +1.$$

(The indices $n_\nu^{(k)}$ mark the beginnings of the k th sign wave in the sequence $\{x_\nu^{(n)}\}$. In the example given above, $n_1^{(1)} = 4$, $n_1^{(2)} = 9$, $n_5^{(2)} = 9$.) For $\nu, \mu = 1, 2, \dots, N$ and $k = 1, 2, \dots$ we put

$$P_\nu^{(k)} = n_\nu^{(k+1)} - n_\nu^{(k)},$$

$$\delta_{\nu\mu}^{(k)} = n_\mu^{(k)} - n_\nu^{(k)}.$$

($P_\nu^{(k)}$ indicates the length of the k th sign wave in the ν th component, and $\delta_{\nu\mu}^{(k)}$ represents the phase difference between the k th sign waves in the ν th and μ th components. In the above example, $P_3^{(1)} = 4$, $\delta_{42}^{(2)} = -2$.) We finally require the quantities

$$\Delta_\nu^{(n)} = [x_\nu^{(n)}]^2 - x_\nu^{(n+1)} x_\nu^{(n-1)}.$$

With these definitions, we can state the following result:

THEOREM. *Let the matrix A satisfy the conditions stated at the beginning of Section 2, and let the vector $x^{(0)}$ be such that $v^H x^{(0)} \neq 0$. Then the vectors $x^{(n)}$ defined by (1) satisfy*

$$(i) \quad \Delta_1^{(n)} : \Delta_2^{(n)} : \dots : \Delta_N^{(n)} \rightarrow r_1^2 : r_2^2 : \dots : r_N^2.$$

For every ν such that $r_\nu \neq 0$, the following two statements hold:

$$(ii) \quad \rho^2 = \lim_{n \rightarrow \infty} \frac{\Delta_\nu^{(n+1)}}{\Delta_\nu^{(n)}};$$

$$(iii) \quad \lim_{k \rightarrow \infty} \frac{P_\nu^{(1)} + P_\nu^{(2)} + \dots + P_\nu^{(k)}}{k} = P$$

exists, and

$$\varphi = \frac{2\pi}{P}.$$

For all ν and μ such that $r_\nu \neq 0$, $r_\mu \neq 0$, the following two statements hold:

(iv) If $\varphi/2\pi$ is irrational, then

$$\lim_{k \rightarrow \infty} \frac{\delta_{\nu\mu}^{(1)} + \delta_{\nu\mu}^{(2)} + \dots + \delta_{\nu\mu}^{(k)}}{kP} \equiv \frac{\varphi_\mu - \varphi_\nu}{2\pi} \pmod{1}$$

(v) If $\varphi/2\pi = p/q$ is rational ($(p, q) = 1$) it can only be asserted that, for some integer l , both the limit superior and the limit inferior of

$$\frac{\delta_{\nu\mu}^{(1)} + \delta_{\nu\mu}^{(2)} + \dots + \delta_{\nu\mu}^{(k)}}{kP}$$

as $k \rightarrow \infty$ differ by at most $1/q$ from

$$\frac{\varphi_\mu - \varphi_\nu}{2\pi} + l.$$

4. Proof of the Theorem. Under the hypotheses of the theorem the matrix A can be represented in the form

$$A = \lambda w^T + \bar{\lambda} \bar{w}^H + A_1$$

where A_1 is a matrix whose eigenvalues lie inside a circle of radius $q\rho$, $0 < q < 1$. If $v^T x^{(0)} = ae^{i\alpha}$, where $a > 0$, we have

$$x^{(n)} = a(\lambda^n u e^{i\alpha} + \bar{\lambda}^n \bar{u} e^{-i\alpha} + w^{(n)})$$

where the components of $w^{(n)}$ are bounded by $Cq^n \rho^n$ with a suitable constant C . Hence

$$(1) \quad x_\nu^{(n)} = 2a\rho^n \{r_\nu \cos(n\varphi + \varphi_\nu + \alpha) + \epsilon_\nu^{(n)}\},$$

where

$$(2) \quad |\epsilon_\nu^{(n)}| \leq Cq^n.$$

A simple calculation now yields

$$\Delta_\nu^{(n)} = \frac{4a^2 \rho^{2n}}{(\sin \varphi)^2} (r_\nu^2 + \eta_\nu^{(n)}),$$

where

$$|\eta_\nu^{(n)}| \leq 2Cq^n \sin^2 \varphi (2r_\nu + Cq^n)$$

and hence $\eta_\nu^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. The relations (i) and, if $r_\nu \neq 0$, (ii) now follow immediately.

For the proof of the remaining statements of the theorem a modified version of (1) is required. Assume the integer n' is such that, for all ν satisfying $r_\nu \neq 0$, $|\epsilon_\nu^{(n)}| < r_\nu$ for $n \geq n'$. Setting temporarily $\beta_\nu^{(n)} = n\varphi + \varphi_\nu + \alpha$, we then have for $n \geq n'$

$$\begin{aligned} (2a\rho^n)^{-1} x_\nu^{(n)} &= \operatorname{Re} \{r_\nu e^{i\beta_\nu^{(n)}} + \epsilon_\nu^{(n)}\} \\ &= \operatorname{Re} \{e^{i\beta_\nu^{(n)}} [r_\nu + e^{-i\beta_\nu^{(n)}} \epsilon_\nu^{(n)}]\} \\ &= |r_\nu + e^{-i\beta_\nu^{(n)}} \epsilon_\nu^{(n)}| \operatorname{Re} \{e^{i(\beta_\nu^{(n)} - \theta_\nu^{(n)})}\}, \end{aligned}$$

where

$$(3) \quad \tan \theta_\nu^{(n)} = \frac{\epsilon_\nu^{(n)} \sin \beta_\nu^{(n)}}{r_\nu + \epsilon_\nu^{(n)} \cos \beta_\nu^{(n)}}, \quad |\theta_\nu^{(n)}| < \frac{\pi}{2}.$$

Hence

$$(4) \quad x_\nu^{(n)} = A_\nu^{(n)} \sin \phi_\nu^{(n)}$$

where

$$\begin{aligned} A_\nu^{(n)} &= 2a\rho^n |r_\nu + e^{-i\beta_\nu^{(n)}} \epsilon_\nu^{(n)}| > 0, \\ \phi_\nu^{(n)} &= n\varphi + \varphi_\nu + \alpha + \frac{\pi}{2} - \theta_\nu^{(n)}. \end{aligned}$$

Formula (4) serves to determine the sign of $x_\nu^{(n)}$ as a function of n .

We shall require an explicit formula for $n_\nu^{(k)}$ valid for large k . It follows from (2) and (3) that $\theta_\nu^{(n)} \rightarrow 0$ for $n \rightarrow \infty$. Let n'' be such that

$$|\theta_\nu^{(n)}| \leq \frac{1}{2} \text{Min}(\varphi, \pi - \varphi), \quad n > n''$$

for all ν satisfying $r_\nu \neq 0$. We then have

$$(5) \quad 0 < \phi_\nu^{(n+1)} - \phi_\nu^{(n)} < \pi, \quad n > n'',$$

i.e., the sequence $\{\phi_\nu^{(n)}\}$ is monotonically increasing and assumes a value in every (open) interval of length π . Let k_ν denote the smallest integer such that

$$n_\nu^{(k_\nu)} > n''.$$

By (5) and by the definition of $n_\nu^{(k)}$, there exists an integer m_ν such that

$$\phi_\nu^{(n^{(k_\nu)})} = n_\nu^{(k_\nu)} \varphi + \varphi_\nu + \alpha + \frac{\pi}{2} - \theta_\nu^{(n)} \geq 2m_\nu \pi,$$

$$\phi_\nu^{(n^{(k_\nu)-1})} = (n_\nu^{(k_\nu)} - 1)\varphi + \varphi_\nu + \alpha + \frac{\pi}{2} - \theta_\nu^{(n-1)} < 2m_\nu \pi.$$

More generally, for $m = 0, 1, 2, \dots$ we have

$$n_\nu^{(k_\nu+m)} \varphi + \varphi_\nu + \alpha + \frac{\pi}{2} - \theta_\nu^{(n)} \geq 2(m_\nu + m)\pi$$

$$(n_\nu^{(k_\nu+m)} - 1)\varphi + \varphi_\nu + \alpha + \frac{\pi}{2} - \theta_\nu^{(n-1)} < 2(m_\nu + m)\pi.$$

We denote, for any real number a , by $[a]$ the largest integer not exceeding a . We also set

$$(6) \quad \psi_\nu = \varphi_\nu + \alpha + \frac{\pi}{2} + 2\pi(k_\nu - m_\nu).$$

If $k = k_\nu + m \geq k_\nu$, it then follows that

$$(7) \quad n_\nu^{(k)} = \left[\frac{2\pi k - \psi_\nu + \theta_\nu^{(n)}}{\varphi} \right].$$

For the proof of (iii) we observe that

$$n_\nu^{(k)} = \frac{2\pi k}{\varphi} + C_\nu^{(k)},$$

where the moduli of the numbers $C_\nu^{(k)}$ are bounded. We have

$$P_\nu^{(1)} + P_\nu^{(2)} + \dots + P_\nu^{(k)} = n_\nu^{(k+1)} - n_\nu^{(1)}$$

and hence, using (7), if $k \geq k_\nu$,

$$\frac{P_\nu^{(1)} + P_\nu^{(2)} + \dots + P_\nu^{(k)}}{k} = \frac{2\pi}{\varphi} + \left(\frac{2\pi}{\varphi} + C_\nu^{(k)} - n_\nu^{(1)} \right) \frac{1}{k}.$$

The second term on the right tends to zero as $k \rightarrow \infty$, and (iii) follows.

For the proof of (iv) we set

$$N_\nu^{(k)} = \frac{1}{2}k - \frac{\varphi}{2\pi k} (n_\nu^{(1)} + n_\nu^{(2)} + \dots + n_\nu^{(k)}).$$

The following lemma is required:

LEMMA. *If $\varphi/2\pi$ is irrational, then there exists a constant c such that for all ν satisfying $r_\nu \neq 0$*

$$\lim_{k \rightarrow \infty} N_\nu^{(k)} \equiv \frac{\varphi_\nu}{2\pi} + c \pmod{1}.$$

The proof consists in showing that for a suitable integer l , and for every $\delta > 0$

$$(8a) \quad \limsup_{k \rightarrow \infty} N_\nu^{(k)} \leq \frac{\varphi_\nu}{2\pi} + c + l + \delta$$

and

$$(8b) \quad \liminf_{k \rightarrow \infty} N_\nu^{(k)} \geq \frac{\varphi_\nu}{2\pi} + c + l - \delta.$$

For $k \geq k_\nu$, let

$$n_\nu^{(k)} = p_\nu^{(k)} + q_\nu^{(k)} + s_\nu^{(k)},$$

where

$$\begin{aligned} p_\nu^{(k)} &= \frac{2\pi k - \psi_\nu}{\varphi}, \\ q_\nu^{(k)} &= \left[\frac{2\pi k - \psi_\nu}{\varphi} \right] - \frac{2\pi k - \varphi_\nu}{\varphi}, \\ s_\nu^{(k)} &= \left[\frac{2\pi k - \psi_\nu + \theta_\nu^{(n)}}{\varphi} \right] - \left[\frac{2\pi k - \psi_\nu}{\varphi} \right]. \end{aligned}$$

Let $\delta > 0$ be given, and let h be an integer such that

$$\left| \frac{\theta^{(n)}}{\varphi} \right| < \delta \quad \text{for } k \geq h,$$

where $n = n_\nu^{(k)}$. We then have

$$N_\nu^{(k)} = \frac{1}{2}k - \frac{1}{2\pi k} \left\{ M + \varphi \sum_{m=h}^k (p_\nu^{(m)} + q_\nu^{(m)} + s_\nu^{(m)}) \right\},$$

where

$$M = (n_\nu^{(1)} + n_\nu^{(2)} + \dots + n_\nu^{(h-1)})\varphi.$$

Since M does not depend on k , $k^{-1}M \rightarrow 0$ as $k \rightarrow \infty$. An easy computation yields

$$\begin{aligned} \frac{1}{2}k - \frac{\varphi}{2\pi k} (p_\nu^{(h)} + p_\nu^{(h+1)} + \dots + p_\nu^{(k)}) \\ = -\frac{1}{2} + \frac{\psi_\nu}{2\pi} + \frac{1}{2\pi k} \{h(h-1) - (h-1)\psi_\nu\}. \end{aligned}$$

The limit of this expression as $k \rightarrow \infty$ exists and equals $-\frac{1}{2} + \psi_\nu/2\pi$.

Since π/φ is irrational, the numbers $q_\nu^{(k)}$ are equidistributed in the interval $(-1, 0]$ according to a classical theorem by H. Weyl (see [6], p. 71, 234). Hence

$$\lim_{k \rightarrow \infty} \frac{1}{k} (q_\nu^{(h)} + q_\nu^{(h+1)} + \dots + q_\nu^{(k)}) = -\frac{1}{2}.$$

According to the definition of h , the numbers $s_\nu^{(k)}$ can differ from 0 only if $q_\nu^{(k)}$ lies either in the interval $(-1, -1 + \delta)$ or in $(-\delta, 0]$. In either case, $|s_\nu^{(k)}| \leq 1$. According to Weyl's theorem, the number of times either possibility occurs is asymptotic to $k\delta$ as $k \rightarrow \infty$. It follows that

$$\limsup_{k \rightarrow \infty} \frac{1}{k} |s_\nu^{(h)} + s_\nu^{(h+1)} + \cdots + s_\nu^{(k)}| \leq \delta.$$

Gathering the above results, we find that

$$\limsup_{k \rightarrow \infty} N_\nu^{(k)} \leq \frac{\psi_\nu}{2\pi} - \frac{1}{2} + \frac{\varphi}{2\pi} (-\frac{1}{2} + \delta),$$

$$\liminf_{k \rightarrow \infty} N_\nu^{(k)} \geq \frac{\psi_\nu}{2\pi} - \frac{1}{2} + \frac{\varphi}{2\pi} (-\frac{1}{2} - \delta).$$

In view of (6), this establishes the relations (8) with

$$c = \frac{\alpha}{2\pi} - \frac{\varphi}{4\pi} - \frac{1}{4}, \quad l_\nu = k_\nu - m_\nu.$$

The statement of the lemma now follows in view of the fact that the above is true for arbitrary $\delta > 0$.

Statement (iv) of the theorem now follows by observing the relation

$$(9) \quad \frac{\delta_{\nu\mu}^{(1)} + \delta_{\nu\mu}^{(2)} + \cdots + \delta_{\nu\mu}^{(k)}}{kP} = N_\mu^{(k)} - N_\nu^{(k)}$$

and letting $k \rightarrow \infty$.

If $2\pi/\varphi = q/p$ is rational and if $(p, q) = 1$, then the numbers $q_\nu^{(k)}$ are no longer equidistributed in $(-1, 0]$, but take on with equal frequency ([4], p. 51) the p distinct values

$$(10) \quad -\frac{m}{p} - \xi, \quad m = 0, 1, \cdots, p-1,$$

where ξ is some number depending on φ , $0 \leq \xi < 1/p$. It follows that

$$(11) \quad \lim_{k \rightarrow \infty} \frac{1}{k} (q_\nu^{(h)} + q_\nu^{(h+1)} + \cdots + q_\nu^{(k)}) = \frac{1-p}{2p} - \xi.$$

If $\xi \neq 0$ in (10), the numbers $s_\nu^{(k)}$ are all zero for k sufficiently large, and thus

$$\lim_{k \rightarrow \infty} \frac{1}{k} (s_\nu^{(h)} + s_\nu^{(h+1)} + \cdots + s_\nu^{(k)}) = 0.$$

If $\xi = 0$, then $s_\nu^{(k)} = -1$ if $q_\nu^{(k)} = 0$ and $\theta_\nu^{(n)} < 0$, and $s_\nu^{(k)} = 0$ otherwise. We have $q_\nu^{(k)} = 0$ every p th time for k large, thus

$$\limsup_{k \rightarrow \infty} \frac{1}{k} (s_\nu^{(h)} + s_\nu^{(h+1)} + \cdots + s_\nu^{(k)}) \leq 0,$$

$$\liminf_{k \rightarrow \infty} \frac{1}{k} (s_\nu^{(h)} + s_\nu^{(h+1)} + \cdots + s_\nu^{(k)}) \geq -\frac{1}{p}.$$

Thus in any case, if $2\pi/\varphi$ is rational,

$$\begin{aligned} \limsup_{k \rightarrow \infty} N_\nu^{(k)} &\cong \frac{\psi_\nu}{2\pi} - \frac{1}{2} - \frac{\varphi}{4\pi} \left(1 - \frac{1}{p}\right), \\ \liminf_{k \rightarrow \infty} N_\nu^{(k)} &\cong \frac{\psi_\nu}{2\pi} - \frac{1}{2} - \frac{\varphi}{4\pi} \left(1 + \frac{1}{p}\right). \end{aligned}$$

From these relations statement (v) of the theorem follows as above by observing (9) and using the relations

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \frac{\delta_{\nu\mu}^{(1)} + \delta_{\nu\mu}^{(2)} + \dots + \delta_{\nu\mu}^{(k)}}{kP} \\ &\leq \limsup_{k \rightarrow \infty} N_\mu^{(k)} - \liminf_{k \rightarrow \infty} N_\nu^{(k)} \\ &\leq \frac{\psi_\mu - \psi_\nu}{2\pi} + \frac{\varphi}{2\pi p} \equiv \frac{\varphi_\mu - \varphi_\nu}{2\pi} + \frac{1}{q} \pmod{1} \end{aligned}$$

and a similar relation for the limit inferior.

5. Numerical Results.

1) As a basis for the numerical experiments we used the following 6×6 matrix A depending on two real parameters a and b , not both 0:

$$A = \alpha UDU^H, \text{ where}$$

$$\alpha = \frac{1}{|a| + |b|}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} \\ \frac{\sqrt{12}}{-1} & \frac{\sqrt{12}}{2} & \frac{\sqrt{12}}{1} & \frac{\sqrt{12}}{1} & \frac{\sqrt{12}}{-1} & \frac{\sqrt{12}}{-2} \\ \frac{\sqrt{12}}{-1} & \frac{\sqrt{12}}{2} & \frac{\sqrt{12}}{1} & \frac{\sqrt{12}}{1} & \frac{\sqrt{12}}{-1} & \frac{\sqrt{12}}{-2} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & -\frac{2}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{\sqrt{8}}{1} & \frac{\sqrt{8}}{-1} & \frac{\sqrt{8}}{-1} & \frac{2}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} \\ \frac{\sqrt{8}}{1} & \frac{\sqrt{8}}{-1} & \frac{\sqrt{8}}{-1} & \frac{2}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} \end{pmatrix}$$

$$D = \left\{ \begin{matrix} a - b & & b^2 & & & \\ -2 & & a + b & & & \\ & & & -4 & & \\ & & & & -2 & \\ & \circ & & & & 1 \\ & & & & & & 3 \end{matrix} \right\}, \text{ and where}$$

U^H is the conjugate transpose of U .

The eigenvalues of A are seen to be:

$$\alpha(a + bi), \quad \alpha(a - bi), \quad -4\alpha, \quad -2\alpha, \quad \alpha, \quad \text{and} \quad 3\alpha.$$

2) For $a = 6$, $b = \frac{1}{2}$, the resulting matrix A was as follows:

$$\begin{bmatrix} 0.2115385 & 0.3653846 & -0.1767767 & -0.05439283 & 0.3386381 & -0.1387861 \\ 0.3653846 & 0.2115385 & 0.1495803 & 0.3807498 & 0.02775722 & -0.005551444 \\ -0.05439283 & 0.2719641 & 0.2307692 & -0.3076923 & 0.2198260 & 0.5338631 \\ -0.1767767 & 0.2583659 & -0.3942308 & 0.5384615 & 0.05103104 & -0.2551552 \\ 0.3386381 & 0.02775722 & 0.1138384 & 0.1570186 & 0.2548077 & -0.1971154 \\ -0.03886011 & 0.09437457 & 0.5691923 & -0.2198260 & -0.1105769 & 0.09134615 \end{bmatrix}$$

For this matrix, the following results were obtained:

Computed		<i>Eigenvalue</i>	
Absolute Value	Computed Argument	Absolute Value	Actual Argument
0.926276	0.083221 radians	0.926276	0.083140 radians

<i>Eigenvector*</i>			
Computed Absolute Value	Computed Argument	Actual Absolute Value	Actual Argument
1	33.45°	1	33.68°
1.00000	33.45°	1.00000	33.68°
0.392232	-90.00°	0.392232	-90.00°
0.980581	52.53°	0.980580	53.12°
0.866025	33.45°	0.866025	33.68°
0.537086	-115.84°	0.537086	-116.57°

3) Other cases:

a) For $a = \frac{1}{2}$ and $b = 5$, the results were as follows:

<i>Eigenvalue</i>			
Computed Absolute Value	Computed Argument	Actual Absolute Value	Actual Argument
0.913625	1.46980 radians	0.913625	1.47113 radians

<i>Eigenvector</i>			
Computed Absolute Value	Computed Argument	Actual Absolute Value	Actual Argument
1	0 radians	1	0 radians
1.00000	0.0000	1.00000	0.0000
1.05267	-0.2815	1.05267	-.2750
0.419137	2.377	0.419137	2.324
0.866025	0.0000	0.866025	0.0000
0.587022	-0.4378	0.587022	-0.4101

b) The case $a = 4$, $b = \frac{1}{2}$ proved to be of interest. Letting $\lambda_1, \dots, \lambda_6$ denote the actual (theoretical) eigenvalues of the matrix corresponding to this case, it turns out that $|\lambda_1| = |\lambda_2| = 0.8958064$ and $|\lambda_3| = 0.8888888$, so that λ_3 is close

* Arguments in 2 were normalized so that the argument of the third component was -90° .

to λ_1 both in location and absolute value. The numerical process for finding the absolute value of λ_1 *did not converge* in this case, *but* the numerical procedure for finding the argument of λ_1 yielded 0.12433 radians as compared with the actual value of 0.12436 radians; i.e., the angle was obtained as accurately in this case as in cases where $|\lambda_1| = |\lambda_2| \gg |\lambda_3| > \dots$.

c) The case $a = 0$, $b = 5$ yielded the eigenvalue (i.e., absolute value and argument) exactly.

NOTE: In each numerical example considered, (1,1,1,1,1) was used as the starting vector, and φ was determined from the average period of *all* components. The latter procedure was found to yield the angle φ more accurately than when the average period of just one component was used.

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