is customarily computed and the correlation matrix is computed from this matrix by using the fact that

$$a_{p+1,p+1} = N,\ a_{i,p+1} = \sum_{n=1}^{N} x_{ni}\,; \quad i = 1, \cdots, p.$$

In addition to adding a component which is identically one to each observation vector, let us form a new vector $c_{n1}, c_{n2}, \cdots, c_{np}$ where c_{ni} is zero if the ith component of the observation vector is missing and one otherwise. Letting each element of missing data have value zero, we form the cross product matrices

$$s_{ij} = \sum_{n=1}^{N} x_{ni}x_{nj} \qquad i, j = 1, \cdots, p + 1$$

$$n_{ij} = \sum_{n=1}^{N} c_{ni}c_{nj} \qquad i, j = 1, \cdots, p.$$

The means m_i, covariances v_{ij}, and correlations r_{ij} are computed from these matrices by the formulas

$$m_i = \frac{1}{n_{ii}} s_{i,p+1}$$

$$v_{ij} = \frac{1}{n_{ij}} s_{ij} - m_i m_j$$

$$r_{ij} = \frac{v_{ij}}{\sqrt{v_{ii}}\sqrt{v_{jj}}}.$$

It should be noted that the statistical properties of these estimates will differ slightly from those computed without missing data. A discussion of some of these properties is given by S. S. Wilks [1].

A FORTRAN program for the computations described in this note is in use at the University of Wisconsin. A write-up and program deck can be obtained by writing to the author.

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1. S. S. Wilks, "Moments and distributions of estimates of population parameters from fragmentary samples," *Ann. Math. Stat.*, v. 3, 1932, p. 163.

Polynomial Approximations to $I_0(x)$, $I_1(x)$ and Related Functions

By F. D. Burgoyne

Hitchcock [1] gives polynomial approximations to some Bessel functions of order zero and one and to some related functions. Notable omissions from his list are any approximations to $I_0(x)$ or $I_1(x)$. The following approximations may serve to fill this gap.

If we write $I_n(x) = (2\pi x)^{-1/2}e^x F_n(x)$, then with the maximum error stated in brackets in each case, and provided $0 \leqq t \leqq 1$,

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$$I_0(4t) = 0.99999\ 99985 + 4.00000\ 01935\, t^2 + 3.99999\ 59541\, t^4$$
$$+ 1.77780\ 99690\, t^6 + 0.44431\ 89384\, t^8 + 0.07137\ 58187\, t^{10}$$
$$+ 0.00759\ 42968\, t^{12} + 0.00082\ 67816\, t^{14}\ (17 \times 10^{-10}),$$

$$t^{-1}I_1(4t) = 1.99999\ 99997 + 4.00000\ 00421\, t^2 + 2.66666\ 57853\, t^4$$
$$+ 0.88889\ 59049\, t^6 + 0.17775\ 04042\, t^8 + 0.02376\ 15011\, t^{10}$$
$$+ 0.00219\ 03549\, t^{12} + 0.00020\ 11611\, t^{14}\ (4 \times 10^{-10}),$$

$$(2\pi)^{-1/2}F_0(4/t) = 0.39894\ 22809 + 0.01246\ 67783\, t + 0.00176\ 23668\, t^2$$
$$+ 0.00026\ 22220\, t^3 + 0.00225\ 85672\, t^4 - 0.01283\ 14822\, t^5$$
$$+ 0.04958\ 11198\, t^6 - 0.12099\ 40805\, t^7 + 0.18954\ 76618\, t^8$$
$$- 0.18677\ 83276\, t^9 + 0.11133\ 15511\, t^{10} - 0.03666\ 94167\, t^{11}$$
$$+ 0.00512\ 46015\, t^{12}\ (7 \times 10^{-10}),$$

$$(2\pi)^{-1/2}F_1(4/t) = 0.39894\ 22799 - 0.03740\ 06642\, t - 0.00293\ 14981\, t^2$$
$$- 0.00043\ 77220\, t^3 - 0.00237\ 87859\, t^4 + 0.01319\ 50213\, t^5$$
$$- 0.05078\ 72951\, t^6 + 0.12301\ 43060\, t^7 - 0.19083\ 32956\, t^8$$
$$+ 0.18552\ 23758\, t^9 - 0.10862\ 98349\, t^{10} + 0.03497\ 54315\, t^{11}$$
$$- 0.00474\ 86397\, t^{12}\ (8 \times 10^{-10}).$$

The first two approximations were obtained by the economization method of Lanczos [2], which is used by Hitchcock. As he notes, this method is inapplicable for the last two approximations, and these were obtained by collocation at the zeros of $T^*_{13}(x) = \cos\{13 \cos^{-1}(2x - 1)\}$.

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1. A. J. M. Hitchcock, "Polynomial approximations to Bessel functions of order zero and one and to related functions," *MTAC*, v. 11, 1957, p. 86–88.
2. C. Lanczos, *Applied Analysis*, Prentice Hall, Inc., New Jersey, 1956.

A Note on the Curve Fitting of Discrete Data by Economization

By F. D. Burgoyne

Suppose that we are given a set of points (x_i, y_i) $0 \leqq i \leqq n$ and we desire to find the polynomial $p(x)$ of given degree $m(<n)$ such that $\max_i |y_i - p(x_i)|$ is a minimum. It is well known that this may be performed in good approximation by using the method of least squares to find the polynomial $q(x)$ of degree m such that $\sum_i \{y_i - q(x_i)\}^2$ is a minimum, and then taking $p(x) = q(x) + c$, where c is constant given by

$$2c = \min_i \{y_i - q(x_i)\} + \max_i \{y_i - q(x_i)\}.$$

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