choice of $\epsilon$. The usual procedure is to divide the range of integration into two parts, integrate outwards for a solution satisfying one boundary condition, integrate inwards for a solution satisfying the other boundary condition, match the solutions at an intermediate point and adjust $\epsilon$ so that the derivatives also agree [1], [2]. The inward integration may be avoided with the procedure described earlier. A convenient way of dividing the range is according to the sign of $f(r)$. For some $r$, $f(r) < 0$ so that condition (ii) is not satisfied: the procedure described here is not always numerically stable when $f(r) < 0$ [3]; in fact, for some values of $i$, $|d_i| < 1$.

Of a series of standard methods, the Numerov method,

$$y_{n+1} = \left( \frac{2 + \frac{h^2}{12} f_{n+1}}{2 + \frac{h^2}{12} f_{n-1}} \right) y_n - \left( 1 - \frac{h^2}{12} f_{n-1} \right) y_{n-1} + \frac{h^2}{12} \left( g_{n+1} + 10g_n + g_{n-1} \right) / \left( 1 - \frac{h^2}{12} f_{n+1} \right),$$

was found to be most accurate in this case, for a given number of evaluations of $f$ for the outward integration. The procedure used successfully was to integrate outwards according to (5) until $f(r) > 0$, then, with the last value computed as a boundary condition, to solve for the “tail” of the wave function by the method described here. The energy adjustment will be the same as before.

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On the Inversion of Sparse Matrices

By A. L. Dulmage and N. S. Mendelsohn

1. Introduction. There are a number of problems in applied mathematics involving many equations in many unknowns, but for which each equation involves only a small fraction of the unknowns. If such problems are linear or are approximated by linearization, one is involved with a matrix, a large proportion of whose entries are zero. To invert such a matrix $A$ it is sometimes advantageous to permute the rows and columns of $A$ yielding $PAQ$ where $P$ and $Q$ are permutation matrices. If

$$PAQ = \begin{bmatrix} A_1 & 0 & & \\ A_2 & & & \\ & \ddots & & \\ * & & & A_r \end{bmatrix}$$

where $A_1, A_2, \cdots, A_r$ are square matrices, the problem of inverting $PAQ$ is reduced

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to inverting the blocks $A_r$ followed by matrix multiplication and addition. On permuting the rows and columns of $(PAQ)^{-1}$ one obtains the matrix $A^{-1}$.

F. Harary [4] gave a method based on the connectivity theory of directed graphs. The blocks $A_i$ turn out to be the matrix representatives of the strong components of an associated directed graph. Harary’s method requires that $Q = P^{-1}$, a restriction which is quite unnecessary for matrix inversion. As a result, many matrices which reduce under independent permutations of rows and columns will not reduce if one insists that $Q = P^{-1}$. To remove this latter restriction, the authors have replaced a directed graph by a bipartite graph. The strong components of a directed graph become the irreducible components of a bipartite graph (see [1] for the definition of an irreducible component of a bipartite graph, [2] for the connection between strong components of directed graphs and irreducible components of bipartite graphs).

2. The Method of Reduction. Let $A$ be a square matrix of order $n$ with entries $a_{ij}$. Associated with $A$ is a bipartite graph $K_A$ with two sets of vertices $S = s_1, s_2, \ldots, s_n$ and $T = t_1, t_2, \ldots, t_n$. A pair $(s_i, t_j)$ is an edge of $K_A$ if and only if $a_{ij} \neq 0$ (one obtains Harary’s directed graph if one identifies the vertices $s_i$ and $t_i$).

Suppose we can find matrices $P$ and $Q$ such that

$$
PAQ = \begin{bmatrix}
A_1 & 0 \\
A_2 & \\
& \\
& A_r
\end{bmatrix}
$$

and such that $A_1, A_2, \ldots, A_r$ cannot be further reduced by permutations of their rows and columns. Then the graphs corresponding to $A_1, A_2, \ldots, A_r$ are the irreducible components of $K_A$.

In [3], the authors give an algorithm for obtaining the irreducible components of a bipartite graph. This algorithm is easily programmable for machine computation and yields the permutations $P$ and $Q$. In the case where $A$ is a non-singular matrix the graph $K_A$ has no tails (see [3] for definition of a tail) and the algorithm described in [3] can be considerably simplified, since the steps needed to isolate and identify the tails can be omitted. An alternative procedure is the following. First, locate a nonzero term in the expansion of the determinant of $A$. This can readily be done using the algorithm of Marshall Hall [5] or Fulkerson and Ford [6]. Next, permute the rows of $A$ until the entries of this term occupy the main diagonal. Call this new matrix $A^*$. Finally, apply the method of F. Harary given in [4] to $A^*$.

We append an example:

$$
\begin{array}{cccccc}
0 & 0 & 5 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 0 \\
0 & 4 & 0 & 3 & 0 & 0 \\
3 & 0 & 1 & 0 & 3 & 2 \\
3 & 2 & 0 & 0 & 2 & 0 \\
1 & 0 & 0 & 4 & 0 & 0 \\
\end{array}
$$

Let $A = \begin{bmatrix}
0 & 0 & 5 & 0 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 2 & 0 & 0 & 1 \\
0 & 4 & 0 & 3 & 0 & 0 & 4 \\
3 & 0 & 1 & 0 & 3 & 2 & 0 \\
3 & 2 & 0 & 0 & 2 & 0 & 3 \\
1 & 0 & 0 & 4 & 0 & 0 & 1 \\
\end{bmatrix}
Note that the corresponding directed graph has the Hamiltonian circuit 1 → 6 → 5 → 3 → 4 → 2 → 7 → 1 and so is strongly connected. Hence, for no permutation \( P \) does \( PAP^{-1} \) reduce. Using the algorithm described in [3] one obtains the permutations \( P = (1 \ 7 \ 2 \ 5 \ 6 \ 4 \ 3) \) and \( Q = (1)(2 \ 3 \ 6 \ 7 \ 5 \ 4) \). Applying \( P \) and \( Q \) to the rows and columns of \( A \), one obtains:

\[
PAQ = \begin{pmatrix}
1 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 4 & 0 & 0 & 0 & 0 \\
3 & 0 & 2 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 4 & 0 & 0 \\
3 & 0 & 0 & 3 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 5 & 1
\end{pmatrix}
\]

The authors are indebted to F. Harary for a pre-publication copy of his paper [4].

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**Missing Data Correlation Computations**

By R. I. Jennrich

In correlation analysis or in any multivariate analysis based on the computation of a correlation or covariance matrix, the applied statistician often runs into the problem of missing data. To avoid complication in computing the correlation matrix, a complete observation vector is often discarded when only one or more of its components are missing. If a correlation matrix is computed by means of a standard electronic computer program, this procedure is often necessary. A large percentage of data may be thrown away when only a small percentage is missing. This note describes a modification in the standard computing scheme which eliminates this waste of data.

Let \( x_{n1}, x_{n2}, \ldots, x_{np} \) denote the \( p \) components of the \( n \)th observation vector, \( n = 1, 2, \ldots, N \). It is customary to add an \( n + 1 \)st component to this vector which is identically equal to one. That is \( x_{n,p+1} = 1 \). The cross product matrix

\[
a_{ij} = \sum_{n=1}^{N} x_{ni}x_{nj} \quad i = 1, \ldots, p + 1; \; j = 1, \ldots, p + 1
\]

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