The process for obtaining a T_ϕ containing μ is now applied to A_2^*. Step 1 of this process immediately gives one element of T_ϕ as element a_{24} of A. Step 1 is completed by deleting row 2, column 4 from A_2^*. This leaves,

$$A_{21}^* = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{32} & 0 \\ a_{41} & 0 & a_{43} \end{pmatrix}.$$

Element a_{41} of this matrix is now set equal to zero as indicated in Step 1. This new form of A_{21}^* satisfies the theorem's hypothesis, so Step 1 is continued by setting a_{43} equal to zero. The matrix A_{21}^* now has the following appearance,

$$A_{21}^* = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The hypothesis of the theorem fails for this matrix and so another element of T_ϕ is a_{43}. Deleting row 3 and column 3 from this last matrix leaves,

$$A_{22}^* = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{32} \end{pmatrix}.$$

Step 1 is repeated on this matrix, and it is seen that T_ϕ contains a_{32} and that $A_{23}^* = (1) = (a_{11})$. From this it follows that the final element of T_ϕ is a_{11}. Therefore, one possible assignment is $T_\phi = \{a_{11}, a_{24}, a_{32}, a_{43}\}$.

As a final remark we note that with obvious simple modifications the algorithm developed here will also solve the analogous problem involving

$$\mu' = \max_{\phi \in \Phi} \min_{a_{ij} \in T_\phi} a_{ij}.$$

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Formulas for Integrals of Products of Associated Legendre or Laguerre Functions

By James Miller

1. Introduction. In this paper we derive, using a very simple technique, formulas for the integrals of products of Legendre functions,

$$(1) \qquad \int_{-1}^{1} P_{n_1}^{m_1}(x) P_{n_2}^{m_2}(x) \cdots P_{n_r}^{m_r}(x)\, dx,$$

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where n_i and m_i are integers and $\sum_{i=1}^{r} m_i$ is even, and of Laguerre functions,

$$\int_0^\infty e^{-\alpha x} x^\beta L_{n_1}^{m_1}(\lambda_1 x) L_{n_2}^{m_2}(\lambda_2 x) \cdots L_{n_r}^{m_r}(\lambda_r x)\, dx \tag{2}$$

where n_i, m_i and β are non-negative integers and $\alpha > 0$.

Gaunt [1] has developed a formula for integral (1) when $r = 3$. Gillis and Weiss [2] and Gillis and Shimshoni [3] give formulas and describe computational methods for the special case of integral (2) where $r = 3$, $\alpha = 3/2$, $\beta = m_1 = m_2 = m_3 = 0$, and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Erdélyi [4] gives a general formula for this integral. Neither the general Erdélyi formula for integral (2) nor the special Gaunt formula for integral (1), however, are particularly well suited for programming for an electronic computer. In view of the importance of special cases of these integrals (in addition to the case treated in references [2] and [3]) in theoretical physics [1, 5, 6], it is desirable to have easily programmed expressions. Moreover, the simplicity of the present approach in developing general formulas for these integrals may itself be of some interest.

2. Formula for the Integral of the Product of Associated Legendre Functions of the First Kind. Taking as the definition of the associated Legendre function

$$P_n^m(x) \equiv \frac{(1 - x^2)^{m/2}}{2^n n!} \frac{d^{m+n}}{dx^{m+n}} (x^2 - 1)^n$$

one can obtain the expression

$$P_n^m(x) = (1 - x^2)^{m/2} \sum_{i=0}^{I(m,n)} c_i(m, n) x^{n-m-2i} \tag{3}$$

where m and n are integers, $n \geqq 0$, and $-n \leqq m \leqq n$;

$$c_i(m, n) = \frac{(-1)^i [2(n - i)]!}{2^n (n - m - 2i)!(n - i)! i!},$$

and $I(m, n) =$ integer part of $\frac{1}{2}(n - m)$. Let

$$P_{n_1 n_2 \cdots n_r}^{m_1 m_2 \cdots m_r} = \int_{-1}^{1} P_{n_1}^{m_1}(x) P_{n_2}^{m_2}(x) \cdots P_{n_r}^{m_r}(x)\, dx$$

and substitute in (3):

$$\begin{aligned} P_{n_1 n_2 \cdots n_r}^{m_1 m_2 \cdots m_r} = \int_{-1}^{1} (1 - x^2)^{M/2} & \sum_{i=0}^{I(n_1,m_1)} \sum_{j=0}^{I(n_2,m_2)} \\ & \cdots \sum_{k=0}^{I(n_r,m_r)} c_i(n_1, m_1) c_j(n_2, m_2) \cdots c_k(n_r, m_r) \cdot x^{N-M-2(i+j+\cdots+k)}\, dx \end{aligned} \tag{4}$$

Here $M = \sum_{i=1}^{r} m_i$, $N = \sum_{i=1}^{r} n_i$.

We wish to expand $(1 - x^2)^{M/2}$ binomially and multiply the resulting polynomial by the other polynomial in the integrand of (4). If the expansion of $(1 - x^2)^{M/2}$ is to terminate, M must be even and non-negative. (For many physics problems of interest, M is even.) If M is negative, we may use the identity

$$P_n^{-m}(x) \equiv (-1)^{|m|} \frac{(n - m)!}{(n + m)!} P_n^{+m}(x)$$

as many times as necessary to render M non-negative. Then

$$P_{n_1 n_2 \cdots n_r}^{m_1 m_2 \cdots m_r} = \int_{-1}^{1} \sum_{q=0}^{M/2} \sum_{i=0}^{I(n_1,m_1)} \sum_{j=0}^{I(n_2,m_2)} \cdots \sum_{k=0}^{I(n_r,m_r)} (-1)^q \cdot \binom{M/2}{q} c_i(n_1, m_1) c_j(n_2, m_2) \cdots c_k(n_r, m_r) \cdot x^{N-M-2(i+j+\cdots+k-q)}\, dx.$$

Integrating gives

$$P_{n_1 n_2 \cdots n_r}^{m_1 m_2 \cdots m_r} = \sum_{q=0}^{M/2} \sum_{i=0}^{I(n_1,m_1)} \sum_{j=0}^{I(n_2,m_2)} \cdots \sum_{k=0}^{I(n_r,m_r)} (-1)^q \cdot \binom{M/2}{q} c_i(n_1, m_1) c_j(n_2, m_2) \cdots c_k(n_r, m_r) \cdot [N - M - 2(i + j + \cdots + k - q) + 1]^{-1} \{1 - (-1)^{[N-M-2(i+j+\cdots+k-q)+1]}\}.$$

This expression vanishes unless the exponent of (-1) is odd. Since

$$-M - 2(i + j + \cdots + k - q) + 1$$

is always odd, N must be even.

Define:

$$\delta(\text{even}, N) = \begin{cases} 1 & \text{if } N \text{ is even} \\ 0 & \text{if } N \text{ is odd.} \end{cases}$$

Then

$$P_{n_1 n_2 \cdots n_r}^{m_1 m_2 \cdots m_r} = 2\delta(\text{even}, N) \sum_{q=0}^{M/2} \sum_{i=0}^{I(n_1,m_1)} \sum_{j=0}^{I(n_2,m_2)} \cdots \sum_{k=0}^{I(n_r,m_r)} (-1)^q \cdot \binom{M/2}{q} c_i(n_1, m_1) c_j(n_2, m_2) \cdots c_k(n_r, m_r) [N - M - 2(i + j + \cdots + k - q) + 1]^{-1}.$$

Programming this expression for a computer may be facilitated by using the relations

$$c_{i+1}(n, m) = -\frac{(n - m - 2i)(n - m - 2i - 1)}{2[2(n - i) - 1](i + 1)} c_i(n, m)$$

$$c_i(n + 1, m) = \frac{2(n - i) + 1}{n - m - 2i + 1} c_i(n, m)$$

$$c_i(n, m + 1) = (n - m - 2i) c_i(n, m).$$

3. Formula for the Integral of the Product of Associated Laguerre Polynomials with Arbitrary Weight Function. Taking as the definition of the associated Laguerre polynomial*

$$L_n^m(x) \equiv \frac{d^m}{dx^m} \left[e^x \frac{d^n}{dx^n} (x^n e^{-x}) \right]$$

* This is the definition usually given in physics books. In mathematical works, the right hand side is multiplied by $\frac{1}{n!}$.

one may obtain the expression

$$L_n^{\,m}(x) = \sum_{i=m}^{n} b_i(n, m)x^{i-m} \tag{5}$$

where $0 \leqq m \leqq n$ and $b_i(n, m) = (-1)^i \dfrac{(n!)^2}{(n-i)!(i-m)!i!}$ where $m \leqq i \leqq n$.

Let

$$L_{n_1 n_2 \cdots n_r}^{m_1 m_2 \cdots m_r} = \int_0^\infty e^{-\alpha x} x^\beta L_{n_1}^{m_1}(\lambda_1 x) L_{n_2}^{m_2}(\lambda_2 x) \cdots L_{n_r}^{m_r}(\lambda_r x)\, dx$$

where $\alpha > 0$ and β is a non-negative integer. Substituting in expression (5) and integrating

$$L_{n_1 n_2 \cdots n_r}^{m_1 m_2 \cdots m_r} = \sum_{i=m_1}^{n_1} \sum_{j=m_2}^{n_2} \cdots \sum_{k=m_r}^{n_r} \lambda_1^{i-m_1} b_i(n_1, m_1) \cdot \lambda_2^{j-m_2} b_j(n_2, m_2)$$

$$\cdots \lambda_r^{k-m_r} b_k(n_r, m_r) \frac{(i+j+\cdots+k+\beta-M)!}{\alpha^{(i+j+\cdots+k+\beta-M+1)}}$$

where $M = \sum_{i=1}^r m_i$. Programming this expression for a computer may be facilitated by using the relations

$$b_{i+1}(n, m) = -\frac{(n-i)}{(i+1)(i-m+1)} \qquad b_i(n, m)$$

$$b_i(n+1, m) = \frac{(n+1)^2}{(n-i+1)} \qquad b_i(n, m)$$

$$b_i(n, m+1) = (i-m) \qquad b_i(n, m).$$

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