

# Matrix Assignments and an Associated Min Max Problem

By T. A. Porsching

**1. Introduction.** Consider an  $n \times n$  matrix,  $A = (a_{ij})$  of positive real numbers, and let  $\Phi$  be the set of  $n!$  permutations of the numbers  $1, 2, \dots, n$ . An assignment  $T_\phi$  is any set  $\{a_{1\phi(1)}, a_{2\phi(2)}, \dots, a_{n\phi(n)}\}$  of  $n$  elements of  $A$  with  $\phi \in \Phi$ . Furthermore, define the number  $\mu$  by the relation,

$$\mu = \min_{\phi \in \Phi} \max_{a_{ij} \in T_\phi} a_{ij}.$$

We are concerned with an algorithm for determining  $\mu$  which is more efficient than the obvious one of generating the  $n!$  possible assignments then straightforwardly selecting  $\mu$ . A method for demonstrating an assignment containing  $\mu$  is also of concern, but such a method is easily evolved using the tools necessary to determine  $\mu$ . Before proceeding we define a *nonzero column* of a set of  $r$  rows of  $A$  as a column which contains at least one nonzero element.

**2. Determination of  $\mu$ .** Note that if  $R$  is the set consisting of the minimum elements of the rows and columns of  $A$ ,  $\mu \geq \mu_0 = \max_{a_{ij} \in R} a_{ij}$ . This is clear if we remember that  $\mu$  is the maximum of some assignment which contains an element from every row and column of  $A$ . In particular, if  $a_{ij}$  is the element of this assignment taken from the  $i$ th row and  $j$ th column of  $A$ ,  $\mu \geq a_{ij} \geq a_{ik}$  where  $a_{ik} \in R$ . The same is true of the  $j$ th column. Since this is true for  $i, j = 1, 2, \dots, n$ ,  $\mu \geq \mu_0$  as asserted. With this in mind we construct an  $n \times n$  matrix  $A_0^*$  which has as its only nonzero elements the elements of  $R$  arranged as they were in  $A$ . If the  $a_{ij}$  are not all distinct, then all elements  $\leq \mu_0$  must also be inserted into  $A_0^*$ . Thus, the matrix  $A_0^*$  is simply the matrix  $A$  with all  $a_{ij}$  such that  $a_{ij} > \mu_0$  replaced by zeros.

Now assume that it is possible to form an assignment from the nonzero elements of  $A_0^*$ . If the maximum element of this assignment is  $\nu$ , then from the definition of  $\mu$ ,  $\mu \leq \nu$ . But  $\nu \leq \mu_0$ , so that  $\mu \leq \nu \leq \mu_0 \leq \mu$ . This implies that  $\mu = \nu = \mu_0$ ; that is,  $\mu$  is the maximum element of  $A_0^*$ .

For the above conclusion it was necessary to assume that an assignment could be formed from the nonzero elements of  $A_0^*$ . Suppose, on the other hand, that every assignment of  $A_0^*$  contains at least one zero. Then clearly  $\mu \geq \mu_1 > \mu_0$ , where  $\mu_1$  is the smallest element of  $A$  greater than  $\mu_0$ . Now alter  $A_0^*$  by inserting in  $A_0^*$  the  $\mu_1$  of  $A$  arranged as they were in  $A$ . This gives a new matrix  $A_1^*$ . The same reasoning used on  $A_0^*$ , shows that if there is an assignment of  $A_1^*$  with no zero elements, then  $\mu = \mu_1$ , the largest element of  $A_1^*$ . In general, it is clear that if  $A_i^*$  is formed by the process of alteration described above, and if  $A_i^*$  is the first such altered matrix which has an assignment containing no zero elements, then  $\mu$

equals the maximum element of  $A_i^*$ . Since  $\mu$  always exists, such an  $A_i^*$  will eventually be found.

Clearly what is lacking in the above method is a relatively simple test on  $A_i^*$  to determine whether or not there exists an assignment of  $A_i^*$  containing no zeros. Fortunately, such a test exists and is essentially described in the following theorem.

**THEOREM 1.** *Let  $A$  be an  $n \times n$  matrix of real numbers. Let  $S_r$  be a set of  $r$  rows of  $A$ . Let  $k$  equal the number of nonzero columns in  $S_r$ . Then there exists an assignment of  $A$  containing no zeros if and only if  $r \leq k$  for all  $S_r$ ,  $r = 1, 2, \dots, n$ .*

We shall prove this theorem by appealing to a more general theorem of Hall on complete systems of distinct representatives (CDR) [1]. Suppose

$$(1) \quad F_1, F_2, \dots, F_m$$

is a finite system of subsets of a given set  $S$ . A CDR of (1) is a set of  $m$  distinct elements of  $S$ :

$$a_1, a_2, \dots, a_m$$

such that  $a_i \in F_i$ . Hall has proven:

**THEOREM 2.** *In order that a CDR of (1) shall exist, it is necessary and sufficient that for each  $k = 1, 2, \dots, m$  any selection of  $k$  of the sets (1) shall contain between them at least  $k$  elements of  $S$ .*

We replace the nonzero elements of  $A$  by integers designating the column in which they lie and let  $S$  be the resulting set of distinct nonzero integers. With  $F_i$  as the set of nonzero integers belonging to the  $i$ th row of the new  $A$ , Theorem 1 follows immediately from Theorem 2.

In view of Theorem 1, the problem now becomes one of generating all of the sets  $S_r$ . This is solved by noting that  $\Sigma$ , the collection of all  $S_r$ , may be put in 1 - 1 correspondence with the set  $\Gamma$  of  $2^n - 1$  distinct, nonvoid combinations of the numbers  $1, 2, \dots, n$ . The correspondence is the obvious one:  $\{n_1, n_2, \dots, n_r\} \in \Gamma \leftrightarrow \{\text{row } n_1, \text{row } n_2, \dots, \text{row } n_r\} \in \Sigma$ . The set  $\Gamma$  is extremely easy to generate on a binary computer since its members correspond in an obvious manner to the binary representation of the numbers  $1, 2, \dots, 2^n - 1$ .

**3. An Assignment for  $\mu$ .** Let  $A_i^*$  be the matrix which yielded  $\mu$ . Then  $A_i^*$  possesses an assignment containing  $\mu$ . Hence, there exists at least one  $\mu$  such that when the row and column containing this  $\mu$  are deleted from  $A_i^*$ , the reduced matrix so obtained,  $A_{i1}^*$ , has an assignment containing no zero elements. The elements of this assignment are the  $n - 1$  remaining elements of the desired assignment. Any element of  $A_{i1}^*$  which does not appear in this assignment will not affect the result of Theorem 1 if set equal to zero. However, if  $A_{i1}^*$  is known to have an assignment containing no zeros and the zeroing of a particular element of  $A_{i1}^*$  implies that the conditions of Theorem 1 do not hold for this matrix, then the zeroed element must be an element of any assignment of  $A_{i1}^*$  which contains no zeros. This gives rise to the following procedure.

1. Sweep  $A_i^*$  setting its nonzero elements equal to zero one at a time, applying Theorem 1 after each zeroing. The first time the conditions of the theorem do not hold, remember the row and column of the last element set equal to zero and delete them from  $A_i^*$  to get  $A_{i1}^*$ .

2. Repeat Step 1 on  $A_{i_1}^*, A_{i_2}^*, \dots, A_{i_{(n-1)}}^*$ . If all remembering is done relative to  $A$ , then the rows and columns so remembered give the positions in  $A$  of  $n$  elements which constitute the desired assignment.

4. **Conclusion.** We conclude with a simple example illustrating the aspects of the preceding development.

For a  $4 \times 4$  matrix the sets  $S_r$  are listed in the following table along with their binary analogs.

Decimal Number	Binary Equivalent	$S_r$ Row Numbers	Decimal Number	Binary Equivalent	$S_r$ Row Numbers
1	0001	1	9	1001	1, 4
2	0010	2	10	1010	2, 4
3	0011	1, 2	11	1011	1, 2, 4
4	0100	3	12	1100	3, 4
5	0101	1, 3	13	1101	1, 3, 4
6	0110	2, 3	14	1110	2, 3, 4
7	0111	1, 2, 3	15	1111	1, 2, 3, 4
8	1000	4			

If

$$A = \begin{pmatrix} 1 & 9 & 4 & 9 \\ 4 & 8 & 2 & 5 \\ 7 & 3 & 7 & 1 \\ 4 & 6 & 3 & 6 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

then

$$A_0^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Since the theorem does not hold for  $S_2 = \{2, 4\}$ ,  $\mu > 3$ . Note that if an  $S_r$  satisfies the condition of the theorem for  $A_{i_r}^*$ , it will also satisfy this condition for  $A_{i_r+j}^*$ ,  $j \geq 0$ . Thus, it is necessary only to consider an  $S_r$  until some  $A_{i_r}^*$  is found which meets the condition of the theorem. In the present example  $A_0^*$  is altered to give,

$$A_1^* = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 4 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \\ 4 & 0 & 3 & 0 \end{pmatrix},$$

and since  $\{1, 2, 4\}$  fails,  $\mu > 4$ . The alteration of  $A_1^*$  yields,

$$A_2^* = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 4 & 0 & 2 & 5 \\ 0 & 3 & 0 & 1 \\ 4 & 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ a_{41} & 0 & a_{43} & 0 \end{pmatrix},$$

for which no  $S_r$  fails and hence  $\mu = 5$ .

The process for obtaining a  $T_\phi$  containing  $\mu$  is now applied to  $A_2^*$ . Step 1 of this process immediately gives one element of  $T_\phi$  as element  $a_{24}$  of  $A$ . Step 1 is completed by deleting row 2, column 4 from  $A_2^*$ . This leaves,

$$A_{21}^* = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{32} & 0 \\ a_{41} & 0 & a_{43} \end{pmatrix}.$$

Element  $a_{41}$  of this matrix is now set equal to zero as indicated in Step 1. This new form of  $A_{21}^*$  satisfies the theorem's hypothesis, so Step 1 is continued by setting  $a_{43}$  equal to zero. The matrix  $A_{21}^*$  now has the following appearance,

$$A_{21}^* = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The hypothesis of the theorem fails for this matrix and so another element of  $T_\phi$  is  $a_{43}$ . Deleting row 3 and column 3 from this last matrix leaves,

$$A_{22}^* = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{32} \end{pmatrix}.$$

Step 1 is repeated on this matrix, and it is seen that  $T_\phi$  contains  $a_{32}$  and that  $A_{23}^* = (1) = (a_{11})$ . From this it follows that the final element of  $T_\phi$  is  $a_{11}$ . Therefore, one possible assignment is  $T_\phi = \{a_{11}, a_{24}, a_{32}, a_{43}\}$ .

As a final remark we note that with obvious simple modifications the algorithm developed here will also solve the analogous problem involving

$$\mu' = \max_{\phi \in \Phi} \min_{a_{ij} \in T_\phi} a_{ij}.$$

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1. P. HALL, "On representatives of subsets," *J. London Math. Soc.* 10, 1935, p. 26-30.

## Formulas for Integrals of Products of Associated Legendre or Laguerre Functions

By James Miller

**1. Introduction.** In this paper we derive, using a very simple technique, formulas for the integrals of products of Legendre functions,

$$(1) \quad \int_{-1}^1 P_{n_1}^{m_1}(x) P_{n_2}^{m_2}(x) \cdots P_{n_r}^{m_r}(x) dx,$$