# Fast Method for Computing the Number of Primes Less than a Given Limit

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**Abstract.** "Fast Method for Computing the Number of Primes Less Than a Given Limit" describes three processes used during the course of calculation. In the first part of the paper the author proves:

$$\phi(x, a) = \phi(x, 1) - \phi\left(\frac{x}{p_2}, 1\right) - \phi\left(\frac{x}{p_3}, 2\right) - \cdots - \phi\left(\frac{x}{p_a}, a - 1\right)$$

where $\phi(x, a)$ represents the number of numbers less than or equal to $x$ and not divisible by the first "$a$" primes. This identity is used to evaluate the formula $\pi(x) = \phi(x, a) + a - 1$, $a + 1 > \pi(\sqrt{x})$ where resulting terms of the form $\phi(x', a')$ are broken down still further by the previously described method, or numerically evaluated using one or both of two other identities, the choice being dependent on $x'$ and $a'$.

Following the paper is a table of calculations made using this process which gives the values of $\pi(x)$ for $x$ at intervals of 10 million up to 1000 million, along with the Riemann and the Chebyshev approximations for $\pi(x)$ and the amount they deviate from the true count.

**1. Definitions and Notations.**

$\pi(x)$: the number of primes less than or equal to $x$.
$p_a$: the $a$th prime, $(p_1 = 2)$.
$m_a$: $p_1 p_2 \cdots p_a$.
$[x]$: greatest integer $\leqq x$.
$c$: any integer.
$c \mid x$: $c$ divides $x$.
$c \nmid x$: $c$ does not divide $x$.
$S(x, a)$: set of all numbers $\leqq x$ and prime to $m_a$.
$\phi(x, a)$: number of members of $S(x, a)$.
$\phi(-x, a)$: equivalent to $-\phi(x, a)$.
$P_s(x, a)$: number of products of $s$ primes belonging to $S(x, a)$, $P_0(x, a) = 1$.
$P_s(-x, a)$: equivalent to $-P_s(x, a)$.
$P_1(x, a)$: by definition is $\pi(x) - a$.

Legendre Sum: $$\phi(x, a) = [x] - \sum_{i \leqq a} [x/p_i] + \sum_{j<i \leqq a} [x/p_i\, p_j] - \sum_{k<j<i \leqq a} [x/p_i\, p_j\, p_k] + \cdots$$

**2. Formula Development.** Let:

$$T_k(x, a) \equiv (-1)^{\beta_0+\beta_1+\cdots+\beta_{a-1}}[x/p_1^{\beta_0} p_2^{\beta_1} \cdots p_a^{\beta_{a-1}}], \tag{1}$$

where $k = 2^{a-1}\beta_{a-1} + 2^{a-2}\beta_{a-2} + \cdots 2^0\beta_0$ such that the $\beta$'s are either 1 or 0, thus

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being the digits of $k$ expressed in binary form and where $0 \leqq k < 2^a$. Let us also adopt: $T_k(-x, a) \equiv -T_k(x, a)$. Legendre's sum may be written in the following order:

$$\phi(x, a) = \sum_{k=0}^{2^a-1} T_k(x, a). \tag{2}$$

Let $M$ be an integer $< 2^a$ and let $2^i$ be the highest power of 2 dividing $M$. Let

$$\gamma(M, x, a) = \sum_{k=M}^{M+2^i-1} T_k(x, a). \tag{3}$$

Full understanding of (3) should be clear from the identity:

$$\phi(x, a) = T_0(x, a) + \gamma(2^0, x, a) + \gamma(2^1, x, a) + \cdots + \gamma(2^{a-1}, x, a), \tag{4}$$

which is (2) with (3) applied.

From (1) it follows that:

$$\{\operatorname{sgn} T_k(x, a)\}_{k>0, 2^i \mid k} = (-1)^{\beta_i + \beta_{i+1} + \cdots + \beta_{a-1}} \tag{5}$$

and

$$| T_k(x, a) |_{k>0, 2^i \mid k} = [x/p_{i+1}^{\beta_i} p_{i+2}^{\beta_{i+1}} \cdots p_a^{\beta_{a-1}}]. \tag{6}$$

By substituting $T_k(x, a)$ for $x$ and $i$ for $a$ in (1) we get:

$$\begin{aligned} T_{k'}\{T_k(x, a), i\} = \operatorname{sgn}\{T_k(x, a)\}(-1)^{\beta_0' + \beta_1' + \cdots + \beta_{i-1}'} \\ \times [| T_k(x, a) | / p_1^{\beta_0'} p_2^{\beta_1'} \cdots p_i^{\beta_{i-1}'}], \end{aligned} \tag{7}$$

where $k' = 2^{i-1}\beta_{i-1}' + 2^{i-2}\beta_{i-2}' + \cdots + 2^0\beta_0'$ and $0 \leqq k'$. When $2^i \mid k$, we can substitute (5) and (6) into (7) giving:

$$T_{k'}\{T_k(x, a), i\}_{2^i \mid k} = T_{k+k'}(x, a). \tag{8}$$

By substituting $T_M(x, a)$ for $x$ and $i$ for $a$ in (2) we get

$$\phi\{T_M(x, a), i\} = \sum_{k'=0}^{2^i-1} T_{k'}\{T_M(x, a), i\}. \tag{9}$$

Using (8), (9) becomes

$$\phi\{T_M(x, a), i\}_{2^i \mid M} = \sum_{k=M}^{M+2^i-1} T_k(x, a)$$

and using (3) we have

$$\gamma(M, x, a) = \phi\{T_M(x, a), i\}. \tag{10}$$

Now (4) becomes

$$\begin{aligned} \phi(x, a) = T_0(x, a) + \phi\{T_{2^0}(x, a), 0\} \\ + \phi\{T_{2^1}(x, a), 1\} + \cdots + \phi\{T_{2^{a-1}}(x, a), a-1\}. \end{aligned} \tag{11}$$

By replacing $x$ with $T_M(x, a)$ and $a$ with $i$, we get

$$\begin{aligned} \phi\{T_M(x, a), i\} = T_0\{T_M(x, a), i\} + \phi(T_{2^0}\{T_M(x, a), i\}, 0) \\ + \phi(T_{2^1}\{T_M(x, a), i\}, 1) + \cdots + \phi(T_{2^{i-1}}\{T_M(x, a), i\}, i-1). \end{aligned} \tag{12}$$

Using (1) to find $T_0\{T_M(x, a), i\}$ and (8) in general, we get

$$\begin{aligned}\phi\{T_M(x, a), i\}_{2^i \mid M} = {} & T_M(x, a) + \phi\{T_{M+2^0}(x, a), 0\} \\ & + \phi\{T_{M+2^1}(x, a), 1\} + \cdots + \phi\{T_{M+2^{i-1}}(x, a), i - 1\}.\end{aligned} \tag{13}$$

We can now calculate $\phi(x, a)$ by (13) setting $\phi(x, a)$ equal to $\phi\{T_0(x, a), a\}$ where $2^a \mid 0$. In (13) $T_M(x, a)$ is computed by (1) while the remaining terms are computed by reapplication of (13) or, if reduced by (1) to a numerical expression of $\phi(x, a)$, by any other method of computing $\phi(x, a)$.

**3. Other Methods of Calculating $\phi$(x, a).** The following method for calculating $\phi(x, a)$ was developed by D. H. Lehmer [1]. Meissel's formula for finding $\pi(x)$ is a special case of this method [1].

$$\phi(x, a) = \sum_{s=0}^{V-1} P_s(x, a), x < p_{a+1}^V \tag{14}$$

where

$$\begin{aligned} P_0(x, a) &= 1, \\ P_1(x, a) &= \pi(x) - a, \\ P_2(x, a) &= \sum_{p_a < p_i \leqq x/p_i} \{\pi(x/p_i) - (i - 1)\}, \\ P_3(x, a) &= \sum_{p_a < p_i \leqq x/p_i^2} \; \sum_{p_i \leqq p_j \leqq x/p_i p_j} \{\pi(x/p_i p_j) - (j - 1)\}. \end{aligned}$$

If $V - 1$ is greater than 3 this method of calculating $\phi(x, a)$ may not be practical. Also

$$\phi(x, a) = c\phi(m_a, a) + \phi(r, a) \tag{15}$$

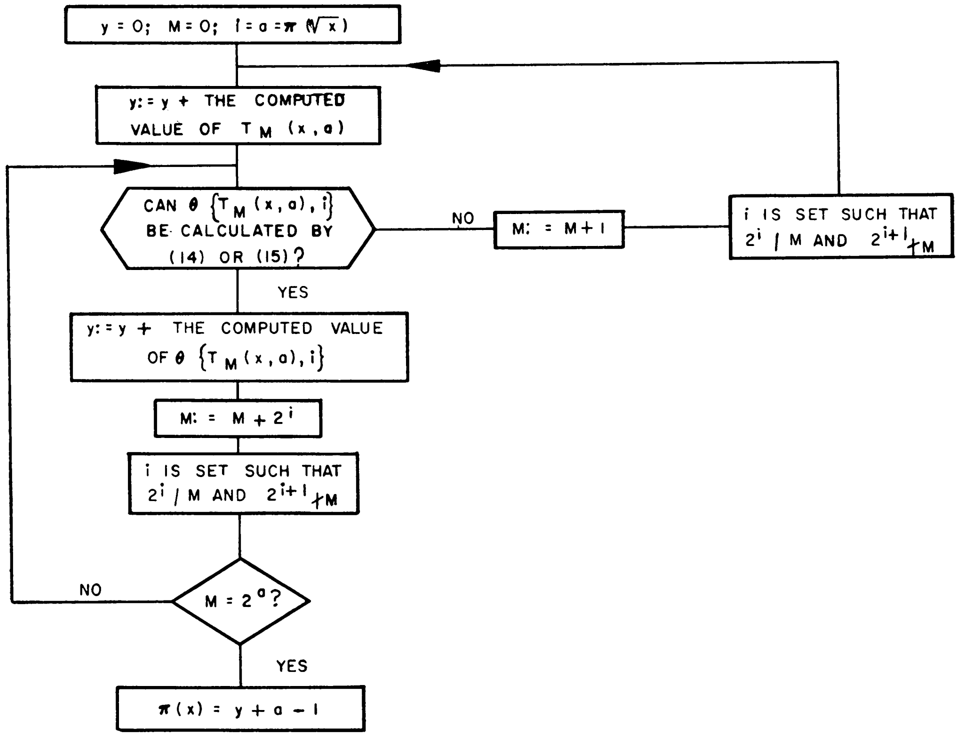
where $x = cm_a + r$, $|r| \leqq \frac{1}{2}m_a$, and $a > 1$. Hence $c\phi(m_a, a)$ can be calculated using a table of the values for $\phi(m_a, a)$ and $\phi(r, a)$ by (14) or (13).

**4. Fast Method for Computing $\pi$(x).** From the above development we can formulate the following procedure for computing $\pi(x)$. Use the formula:

$$\pi(x) = \phi(x, a) + a - 1 \tag{16}$$

where $p_{a+1}^2 \geqq x \geqq p_a^2$. Place a limit on $V$ in (14) such that $\phi(x, a)$ is not computable by (14) when $p_{a+1}^4 \leqq x$. Now place a limit on $a$ in (15) such that $\phi(r, a)$ will always be computable by (14) with its restriction $V \leqq 4$. This means the value of $a$ in (15) must meet the requirement $1 < a \leqq 6$. With a limited table of primes, the larger the better, we are prepared to compute $\pi(x)$.

Compute $\phi(x, a)$ in (16) by (13) where the resulting term $T_0(x, a)$ is computed by (1). The rest of the resulting terms from (13) considered as expressions of the form $\phi(x, a)$ are computed by (1) if $a = 1$, by (15), $\phi(r, a)$ being computed by (14), if $1 < a \leqq 6$, and by (14) or the reapplication of (13) if $a > 6$. In the latter case $\phi(x, a)$ is computed by (14) unless $V > 4$ or $\pi(x)$ is too large to be found in the table of primes when $2 \leqq V \leqq 4$. The resulting terms from a reapplication of (13) are calculated using the process used in calculating the resulting terms of the calculation of $\phi(x, a)$ in (16).




We now introduce a system of order as an aid to our method of calculating $\pi(x)$. At any point in our calculations we will have a number of terms calculated. Introducing $y$ as the sum of these calculated terms, here is the system of order in flow chart form.

The length of time necessary for the calculation of $\pi(x)$ on the 709 using the author's program was:

$$t \sim 60x^{\log_{10} 5} \mu\text{sec}.$$

In the following table the "$Li$" function is Chebyshev's approximation for $\pi(x)$ and the "$R$" function is the Riemann approximation

$$Li(x) = \int_2^x \frac{dt}{\ln t},$$

$$R(x) = \sum_{n=1}^{\infty} n^{-1}\mu(n)Li(x^{1/n}).$$

They are computed using

$$Li(x) = \gamma + \ln(\ln x) + \ln x + \frac{(\ln x)^2}{2 \times 2!} + \frac{(\ln x)^3}{3 \times 3!} + \cdots + \frac{(\ln x)^{75}}{75 \times 75!},$$

$$R(x) = 1 + \frac{\ln x}{S_2} + \frac{(\ln x)^2}{S_3(2 \times 2!)} + \cdots + \frac{(\ln x)^{75}}{S_{76}(75 \times 75!)}$$

where $S_n = \sum_1^\infty k^{-n}$.

Although the table gives $x$ only at intervals of 10 million, calculations have also been made for $x$ at intervals of 1 million, with the results listed in the same manner. Values of $\pi(x)$ for $x$ (in millions) at 1, 2, ... , 10 checked with values obtained by D. N. Lehmer [2]. The values of $\pi(x)$ for the following values of $x$ (in millions) checked with those listed by D. H. Lehmer [1, p. 386]: 20, 25, 33, 37, 40, 90, 100, 999, and 1000.

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1. D. H. LEHMER, *Illinois J. Math.*, v. 3, no. 3, p. 381–388, 1959.
2. D. N. LEHMER, *List of Prime Numbers from 1 to 10,006,721*, New York, Hafner Pub. Co., 1956.

*Comparison of the Count of Primes with the Corresponding Values of the Formulas of Chebyshev and Riemann at intervals of 10,000,000 up to 1,000,000,000.*

| $x$ | $\pi(x)$ | $Li(x)$ | $R(x)$ | $Li(x) - \pi(x)$ | $R(x) - \pi(x)$ |
|---|---|---|---|---|---|
| 10000000 | 664579 | 664918 | 664667 | +339 | +88 |
| 20000000 | 1270607 | 1270905 | 1270571 | +298 | −36 |
| 30000000 | 1857859 | 1858213 | 1857818 | +354 | −41 |
| 40000000 | 2433654 | 2434016 | 2433570 | +362 | −84 |
| 50000000 | 3001134 | 3001557 | 3001067 | +423 | −67 |
| 60000000 | 3562115 | 3562683 | 3562154 | +568 | +39 |
| 70000000 | 4118064 | 4118585 | 4118020 | +521 | −44 |
| 80000000 | 4669382 | 4670091 | 4669493 | +709 | +111 |
| 90000000 | 5216954 | 5217810 | 5217182 | +856 | +228 |
| 100000000 | 5761455 | 5762209 | 5761552 | +754 | +97 |
| 110000000 | 6303309 | 6303656 | 6302971 | +347 | −338 |
| 120000000 | 6841648 | 6842446 | 6841735 | +798 | +87 |
| 130000000 | 7378187 | 7378825 | 7378090 | +638 | −97 |
| 140000000 | 7912199 | 7912998 | 7912239 | +799 | +40 |
| 150000000 | 8444396 | 8445139 | 8444357 | +743 | −39 |
| 160000000 | 8974458 | 8975397 | 8974593 | +939 | +135 |
| 170000000 | 9503083 | 9503902 | 9503077 | +819 | −6 |
| 180000000 | 10030385 | 10030768 | 10029923 | +383 | −462 |
| 190000000 | 10555473 | 10556096 | 10555231 | +623 | −242 |
| 200000000 | 11078937 | 11079975 | 11079090 | +1038 | +153 |
| 210000000 | 11601626 | 11602484 | 11601581 | +858 | −45 |
| 220000000 | 12122540 | 12123696 | 12122775 | +1156 | +235 |
| 230000000 | 12642573 | 12643676 | 12642736 | +1103 | +163 |
| 240000000 | 13161544 | 13162482 | 13161525 | +938 | −19 |
| 250000000 | 13679318 | 13680169 | 13679195 | +851 | −123 |
| 260000000 | 14195860 | 14196786 | 14195796 | +926 | −64 |
| 270000000 | 14711384 | 14712378 | 14711372 | +994 | −12 |
| 280000000 | 15226069 | 15226988 | 15225965 | +919 | −104 |
| 290000000 | 15739663 | 15740653 | 15739614 | +990 | −49 |
| 300000000 | 16252325 | 16253409 | 16252355 | +1084 | +30 |
| 310000000 | 16764521 | 16765291 | 16764222 | +770 | −299 |
| 320000000 | 17275206 | 17276328 | 17275245 | +1122 | +39 |
| 330000000 | 17785475 | 17786551 | 17785453 | +1076 | −22 |
| 340000000 | 18294605 | 18295985 | 18294873 | +1380 | +268 |
| 350000000 | 18803526 | 18804658 | 18803531 | +1132 | +5 |
| 360000000 | 19311288 | 19312592 | 19311452 | +1304 | +164 |
| 370000000 | 19818405 | 19819810 | 19818656 | +1405 | +251 |
| 380000000 | 20325373 | 20326334 | 20325167 | +961 | −206 |
| 390000000 | 20831210 | 20832184 | 20831003 | +974 | −207 |
| 400000000 | 21336326 | 21337378 | 21336185 | +1052 | −141 |
| 410000000 | 21840713 | 21841935 | 21840729 | +1222 | +16 |
| 420000000 | 22344479 | 22345872 | 22344653 | +1393 | +174 |
| 430000000 | 22848050 | 22849204 | 22847973 | +1154 | −77 |
| 440000000 | 23350555 | 23351948 | 23350705 | +1393 | +150 |
| 450000000 | 23853038 | 23854119 | 23852863 | +1081 | −175 |
| 460000000 | 24354548 | 24355729 | 24354461 | +1181 | −87 |
| 470000000 | 24855718 | 24856793 | 24855513 | +1075 | −205 |
| 480000000 | 25356424 | 25357324 | 25356032 | +900 | −392 |
| 490000000 | 25856368 | 25857333 | 25856029 | +965 | −339 |
| 500000000 | 26355867 | 26356832 | 26355517 | +965 | −350 |

| $x$ | $\pi(x)$ | $Li(x)$ | $R(x)$ | $Li(x) - \pi(x)$ | $R(x) - \pi(x)$ |
|---|---|---|---|---|---|
| 510000000 | 26854252 | 26855833 | 26854507 | +1581 | +255 |
| 520000000 | 27352687 | 27354346 | 27353008 | +1659 | +321 |
| 530000000 | 27850698 | 27852381 | 27851033 | +1683 | +335 |
| 540000000 | 28348381 | 28349949 | 28348589 | +1568 | +208 |
| 550000000 | 28845356 | 28847059 | 28845688 | +1703 | +332 |
| 560000000 | 29342150 | 29343720 | 29342338 | +1570 | +188 |
| 570000000 | 29838286 | 29839940 | 29838548 | +1654 | +262 |
| 580000000 | 30334175 | 30335730 | 30334327 | +1555 | +152 |
| 590000000 | 30829544 | 30831095 | 30829682 | +1551 | +138 |
| 600000000 | 31324703 | 31326045 | 31324622 | +1342 | −81 |
| 610000000 | 31819444 | 31820587 | 31819153 | +1143 | −291 |
| 620000000 | 32313388 | 32314729 | 32313285 | +1341 | −103 |
| 630000000 | 32807229 | 32808477 | 32807023 | +1248 | −206 |
| 640000000 | 33300450 | 33301838 | 33300374 | +1388 | −76 |
| 650000000 | 33793395 | 33794819 | 33793345 | +1424 | −50 |
| 660000000 | 34286170 | 34287427 | 34285943 | +1257 | −227 |
| 670000000 | 34778319 | 34779667 | 34778173 | +1348 | −146 |
| 680000000 | 35270167 | 35271546 | 35270042 | +1379 | −125 |
| 690000000 | 35761747 | 35763069 | 35761556 | +1322 | −191 |
| 700000000 | 36252931 | 36254242 | 36252719 | +1311 | −212 |
| 710000000 | 36743905 | 36745071 | 36743539 | +1166 | −366 |
| 720000000 | 37234048 | 37235561 | 37234019 | +1513 | −29 |
| 730000000 | 37724170 | 37725717 | 37724166 | +1547 | −4 |
| 740000000 | 38213987 | 38215544 | 38213984 | +1557 | −3 |
| 750000000 | 38703181 | 38705046 | 38703477 | +1865 | +296 |
| 760000000 | 39192219 | 39194230 | 39192652 | +2011 | +433 |
| 770000000 | 39680979 | 39683099 | 39681512 | +2120 | +533 |
| 780000000 | 40169476 | 40171658 | 40170062 | +2182 | +586 |
| 790000000 | 40658253 | 40659911 | 40658306 | +1658 | +53 |
| 800000000 | 41146179 | 41147862 | 41146248 | +1683 | +69 |
| 810000000 | 41634187 | 41635516 | 41633893 | +1329 | −294 |
| 820000000 | 42121502 | 42122877 | 42121245 | +1375 | −257 |
| 830000000 | 42608404 | 42609948 | 42608308 | +1544 | −96 |
| 840000000 | 43095410 | 43096733 | 43095084 | +1323 | −326 |
| 850000000 | 43581966 | 43583236 | 43581579 | +1270 | −387 |
| 860000000 | 44067840 | 44069462 | 44067796 | +1622 | −44 |
| 870000000 | 44553888 | 44555412 | 44553738 | +1524 | −150 |
| 880000000 | 45039361 | 45041091 | 45039408 | +1730 | +47 |
| 890000000 | 45524412 | 45526502 | 45524811 | +2090 | +399 |
| 900000000 | 46009215 | 46011649 | 46009949 | +2434 | +734 |
| 910000000 | 46494557 | 46496534 | 46494826 | +1977 | +269 |
| 920000000 | 46979583 | 46981161 | 46979445 | +1578 | −138 |
| 930000000 | 47463433 | 47465532 | 47463809 | +2099 | +376 |
| 940000000 | 47947424 | 47949652 | 47947920 | +2228 | +496 |
| 950000000 | 48431471 | 48433523 | 48431783 | +2052 | +312 |
| 960000000 | 48915316 | 48917147 | 48915399 | +1831 | +83 |
| 970000000 | 49398798 | 49400527 | 49398771 | +1729 | −27 |
| 980000000 | 49881580 | 49883667 | 49881903 | +2087 | +323 |
| 990000000 | 50364709 | 50366569 | 50364797 | +1860 | +88 |
| 1000000000 | 50847534 | 50849235 | 50847455 | +1701 | −79 |