Fast Method for Computing the Number of Primes Less than a Given Limit

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Abstract. "Fast Method for Computing the Number of Primes Less Than a Given Limit" describes three processes used during the course of calculation. In the first part of the paper the author proves:

$$\phi(x, a) = \phi(x, 1) - \phi\left(\frac{x}{p_2}, 1\right) - \phi\left(\frac{x}{p_3}, 2\right) - \cdots - \phi\left(\frac{x}{p_a}, a - 1\right)$$

where $\phi(x, a)$ represents the number of numbers less than or equal to x and not divisible by the first "a" primes. This identity is used to evaluate the formula $\pi(x) = \phi(x, a) + a - 1, \quad a + 1 > \pi(\sqrt{x})$ where resulting terms of the form $\phi(x', a')$ are broken down still further by the previously described method, or numerically evaluated using one or both of two other identities, the choice being dependent on x' and a'.

Following the paper is a table of calculations made using this process which gives the values of $\pi(x)$ for x at intervals of 10 million up to 1000 million, along with the Riemann and the Chebyshev approximations for $\pi(x)$ and the amount they deviate from the true count.

1. Definitions and Notations.

$\pi(x)$: the number of primes less than or equal to x.
p_a: the ath prime, ($p_1 = 2$).
m_a: $p_1 p_2 \cdots p_a$.
$[x]$: greatest integer $\leqq x$.
c: any integer.
$c \mid x$: c divides x.
$c \nmid x$: c does not divide x.
$S(x, a)$: set of all numbers $\leqq x$ and prime to m_a.
$\phi(x, a)$: number of members of $S(x, a)$.
$\phi(-x, a)$: equivalent to $-\phi(x, a)$.
$P_s(x, a)$: number of products of s primes belonging to $S(x, a)$, $P_0(x, a) = 1$.
$P_s(-x, a)$: equivalent to $-P_s(x, a)$.
$P_1(x, a)$: by definition is $\pi(x) - a$.

Legendre Sum:

$$\phi(x, a) = [x] - \sum_{i \leqq a} [x/p_i] + \sum_{j<i \leqq a} [x/p_i\, p_j] - \sum_{k<j<i \leqq a} [x/p_i\, p_j\, p_k] + \cdots$$

2. Formula Development.

Let:

$$T_k(x, a) \equiv (-1)^{\beta_0+\beta_1+\cdots+\beta_{a-1}}[x/p_1^{\beta_0} p_2^{\beta_1} \cdots p_a^{\beta_{a-1}}], \tag{1}$$

where $k = 2^{a-1}\beta_{a-1} + 2^{a-2}\beta_{a-2} + \cdots 2^0\beta_0$ such that the β's are either 1 or 0, thus

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being the digits of k expressed in binary form and where $0 \leqq k < 2^a$. Let us also adopt: $T_k(-x, a) \equiv -T_k(x, a)$. Legendre's sum may be written in the following order:

$$\phi(x, a) = \sum_{k=0}^{2^a-1} T_k(x, a). \tag{2}$$

Let M be an integer $< 2^a$ and let 2^i be the highest power of 2 dividing M. Let

$$\gamma(M, x, a) = \sum_{k=M}^{M+2^i-1} T_k(x, a). \tag{3}$$

Full understanding of (3) should be clear from the identity:

$$\phi(x, a) = T_0(x, a) + \gamma(2^0, x, a) + \gamma(2^1, x, a) + \cdots + \gamma(2^{a-1}, x, a), \tag{4}$$

which is (2) with (3) applied.

From (1) it follows that:

$$\{\operatorname{sgn} T_k(x, a)\}_{k>0,2^i|k} = (-1)^{\beta_i+\beta_{i+1}+\cdots+\beta_{a-1}} \tag{5}$$

and

$$| T_k(x, a) |_{k>0,2^i|k} = [x/p_{i+1}^{\beta_i} p_{i+2}^{\beta_{i+1}} \cdots p_a^{\beta_{a-1}}]. \tag{6}$$

By substituting $T_k(x, a)$ for x and i for a in (1) we get:

$$\begin{aligned} T_{k'}\{T_k(x, a), i\} = \operatorname{sgn}\{T_k(x, a)\}&(-1)^{\beta_0'+\beta_1'+\cdots+\beta_{i-1}'} \\ &\times [| T_k(x, a) | /p_1^{\beta_0'} p_2^{\beta_1'} \cdots p_i^{\beta_{i-1}'}], \end{aligned} \tag{7}$$

where $k' = 2^{i-1}\beta_{i-1}' + 2^{i-2}\beta_{i-2}' + \cdots + 2^0\beta_0'$ and $0 \leqq k'$. When $2^i \mid k$, we can substitute (5) and (6) into (7) giving:

$$T_{k'}\{T_k(x, a), i\}_{2^i|k} = T_{k+k'}(x, a). \tag{8}$$

By substituting $T_M(x, a)$ for x and i for a in (2) we get

$$\phi\{T_M(x, a), i\} = \sum_{k'=0}^{2^i-1} T_{k'}\{T_M(x, a), i\}. \tag{9}$$

Using (8), (9) becomes

$$\phi\{T_M(x, a), i\}_{2^i|M} = \sum_{k=M}^{M+2^i-1} T_k(x, a)$$

and using (3) we have

$$\gamma(M, x, a) = \phi\{T_M(x, a), i\}. \tag{10}$$

Now (4) becomes

$$\begin{aligned} \phi(x, a) = T_0(x, a) &+ \phi\{T_{2^0}(x, a), 0\} \\ &+ \phi\{T_{2^i}(x, a), 1\} + \cdots + \phi\{T_{2^{a-1}}(x, a), a-1\}. \end{aligned} \tag{11}$$

By replacing x with $T_M(x, a)$ and a with i, we get

$$\begin{aligned} \phi\{T_M(x, a), i\} = T_0\{T_M(x, a), i\} &+ \phi(T_{2^0}\{T_M(x, a), i\}, 0) \\ &+ \phi(T_{2^1}\{T_M(x, a), i\}, 1) + \cdots + \phi(T_{2^{i-1}}\{T_M(x, a), i\}, i-1). \end{aligned} \tag{12}$$

Using (1) to find $T_0\{T_M(x, a), i\}$ and (8) in general, we get

$$\phi\{T_M(x, a), i\}_{2^i | M} = T_M(x, a) + \phi\{T_{M+2^0}(x, a), 0\} + \phi\{T_{M+2^1}(x, a), 1\} + \cdots + \phi\{T_{M+2^{i-1}}(x, a), i - 1\}. \tag{13}$$

We can now calculate $\phi(x, a)$ by (13) setting $\phi(x, a)$ equal to $\phi\{T_0(x, a), a\}$ where $2^a \mid 0$. In (13) $T_M(x, a)$ is computed by (1) while the remaining terms are computed by reapplication of (13) or, if reduced by (1) to a numerical expression of $\phi(x, a)$, by any other method of computing $\phi(x, a)$.

3. Other Methods of Calculating ϕ(x, a). The following method for calculating $\phi(x, a)$ was developed by D. H. Lehmer [1]. Meissel's formula for finding $\pi(x)$ is a special case of this method [1].

$$\phi(x, a) = \sum_{s=0}^{V-1} P_s(x, a), x < p_{a+1}^V \tag{14}$$

where

$$P_0(x, a) = 1,$$

$$P_1(x, a) = \pi(x) - a,$$

$$P_2(x, a) = \sum_{p_a < p_i \leqq x/p_i} \{\pi(x/p_i) - (i - 1)\},$$

$$P_3(x, a) = \sum_{p_a < p_i \leqq x/p_i^2} \; \sum_{p_i \leqq p_j \leqq x/p_i p_j} \{\pi(x/p_i p_j) - (j - 1)\}.$$

If $V - 1$ is greater than 3 this method of calculating $\phi(x, a)$ may not be practical. Also

$$\phi(x, a) = c\phi(m_a, a) + \phi(r, a) \tag{15}$$

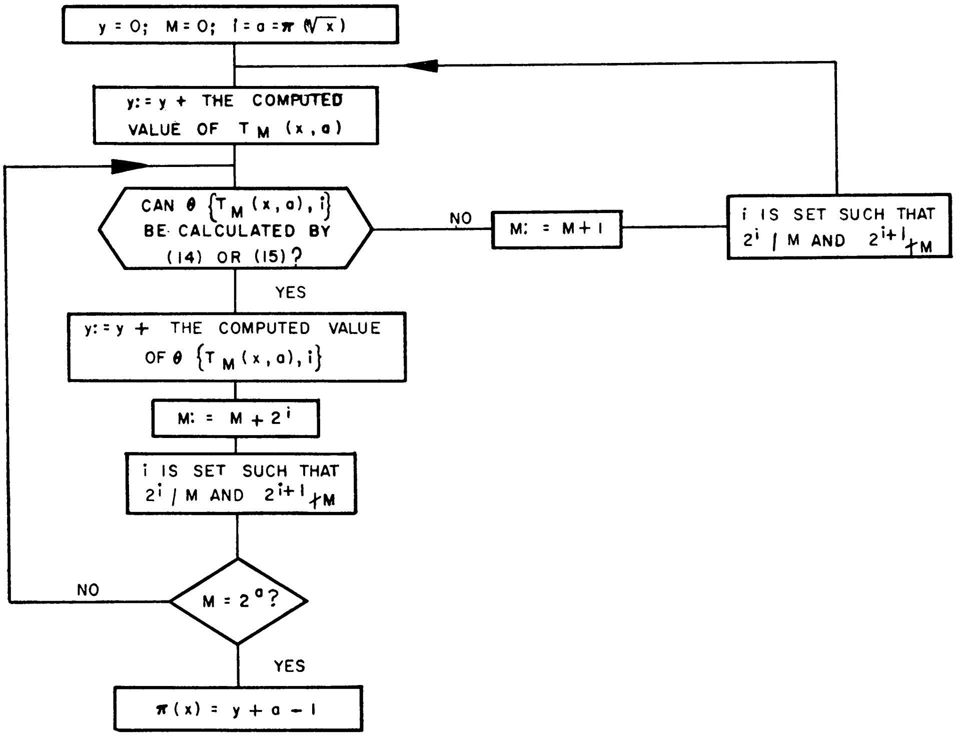
where $x = cm_a + r$, $|r| \leqq \frac{1}{2}m_a$, and $a > 1$. Hence $c\phi(m_a, a)$ can be calculated using a table of the values for $\phi(m_a, a)$ and $\phi(r, a)$ by (14) or (13).

4. Fast Method for Computing π(x). From the above development we can formulate the following procedure for computing $\pi(x)$. Use the formula:

$$\pi(x) = \phi(x, a) + a - 1 \tag{16}$$

where $p_{a+1}^2 \geqq x \geqq p_a^2$. Place a limit on V in (14) such that $\phi(x, a)$ is not computable by (14) when $p_{a+1}^4 \leqq x$. Now place a limit on a in (15) such that $\phi(r, a)$ will always be computable by (14) with its restriction $V \leqq 4$. This means the value of a in (15) must meet the requirement $1 < a \leqq 6$. With a limited table of primes, the larger the better, we are prepared to compute $\pi(x)$.

Compute $\phi(x, a)$ in (16) by (13) where the resulting term $T_0(x, a)$ is computed by (1). The rest of the resulting terms from (13) considered as expressions of the form $\phi(x, a)$ are computed by (1) if $a = 1$, by (15), $\phi(r, a)$ being computed by (14), if $1 < a \leqq 6$, and by (14) or the reapplication of (13) if $a > 6$. In the latter case $\phi(x, a)$ is computed by (14) unless $V > 4$ or $\pi(x)$ is too large to be found in the table of primes when $2 \leqq V \leqq 4$. The resulting terms from a reapplication of (13) are calculated using the process used in calculating the resulting terms of the calculation of $\phi(x, a)$ in (16).



We now introduce a system of order as an aid to our method of calculating $\pi(x)$. At any point in our calculations we will have a number of terms calculated. Introducing y as the sum of these calculated terms, here is the system of order in flow chart form.

The length of time necessary for the calculation of $\pi(x)$ on the 709 using the author's program was:

$$t \sim 60x^{\log_{10} 5}\mu\text{sec}.$$

In the following table the "Li" function is Chebyshev's approximation for $\pi(x)$ and the "R" function is the Riemann approximation

$$Li(x) = \int_2^x \frac{dt}{\ln t},$$

$$R(x) = \sum_{n=1}^{\infty} n^{-1}\mu(n)Li(x^{1/n}).$$

They are computed using

$$Li(x) = \gamma + \ln(\ln x) + \ln x + \frac{(\ln x)^2}{2 \times 2!} + \frac{(\ln x)^3}{3 \times 3!} + \cdots + \frac{(\ln x)^{75}}{75 \times 75!},$$

$$R(x) = 1 + \frac{\ln x}{S_2} + \frac{(\ln x)^2}{S_3(2 \times 2!)} + \cdots + \frac{(\ln x)^{75}}{S_{76}(75 \times 75!)}$$

where $S_n = \sum_1^{\infty} k^{-n}$.

Although the table gives x only at intervals of 10 million, calculations have also been made for x at intervals of 1 million, with the results listed in the same manner. Values of $\pi(x)$ for x (in millions) at 1, 2, ... , 10 checked with values obtained by D. N. Lehmer [2]. The values of $\pi(x)$ for the following values of x (in millions) checked with those listed by D. H. Lehmer [1, p. 386]: 20, 25, 33, 37, 40, 90, 100, 999, and 1000.

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1. D. H. Lehmer, *Illinois J. Math.*, v. 3, no. 3, p. 381–388, 1959.
2. D. N. Lehmer, *List of Prime Numbers from 1 to 10,006,721*, New York, Hafner Pub. Co., 1956.

Comparison of the Count of Primes with the Corresponding Values of the Formulas of Chebyshev and Riemann at intervals of 10,000,000 up to 1,000,000,000.

x	$\pi(x)$	$Li(x)$	$R(x)$	$Li(x) - \pi(x)$	$R(x) - \pi(x)$
10000000	664579	664918	664667	+339	+88
20000000	1270607	1270905	1270571	+298	−36
30000000	1857859	1858213	1857818	+354	−41
40000000	2433654	2434016	2433570	+362	−84
50000000	3001134	3001557	3001067	+423	−67
60000000	3562115	3562683	3562154	+568	+39
70000000	4118064	4118585	4118020	+521	−44
80000000	4669382	4670091	4669493	+709	+111
90000000	5216954	5217810	5217182	+856	+228
100000000	5761455	5762209	5761552	+754	+97
110000000	6303309	6303656	6302971	+347	−338
120000000	6841648	6842446	6841735	+798	+87
130000000	7378187	7378825	7378090	+638	−97
140000000	7912199	7912998	7912239	+799	+40
150000000	8444396	8445139	8444357	+743	−39
160000000	8974458	8975397	8974593	+939	+135
170000000	9503083	9503902	9503077	+819	−6
180000000	10030385	10030768	10029923	+383	−462
190000000	10555473	10556096	10555231	+623	−242
200000000	11078937	11079975	11079090	+1038	+153
210000000	11601626	11602484	11601581	+858	−45
220000000	12122540	12123696	12122775	+1156	+235
230000000	12642573	12643676	12642736	+1103	+163
240000000	13161544	13162482	13161525	+938	−19
250000000	13679318	13680169	13679195	+851	−123
260000000	14195860	14196786	14195796	+926	−64
270000000	14711384	14712378	14711372	+994	−12
280000000	15226069	15226988	15225965	+919	−104
290000000	15739663	15740653	15739614	+990	−49
300000000	16252325	16253409	16252355	+1084	+30
310000000	16764521	16765291	16764222	+770	−299
320000000	17275206	17276328	17275245	+1122	+39
330000000	17785475	17786551	17785453	+1076	−22
340000000	18294605	18295985	18294873	+1380	+268
350000000	18803526	18804658	18803531	+1132	+5
360000000	19311288	19312592	19311452	+1304	+164
370000000	19818405	19819810	19818656	+1405	+251
380000000	20325373	20326334	20325167	+961	−206
390000000	20831210	20832184	20831003	+974	−207
400000000	21336326	21337378	21336185	+1052	−141
410000000	21840713	21841935	21840729	+1222	+16
420000000	22344479	22345872	22344653	+1393	+174
430000000	22848050	22849204	22847973	+1154	−77
440000000	23350555	23351948	23350705	+1393	+150
450000000	23853038	23854119	23852863	+1081	−175
460000000	24354548	24355729	24354461	+1181	−87
470000000	24855718	24856793	24855513	+1075	−205
480000000	25356424	25357324	25356032	+900	−392
490000000	25856368	25857333	25856029	+965	−339
500000000	26355867	26356832	26355517	+965	−350

x	$\pi(x)$	$Li(x)$	$R(x)$	$Li(x) - \pi(x)$	$R(x) - \pi(x)$
510000000	26854252	26855833	26854507	+1581	+255
520000000	27352687	27354346	27353008	+1659	+321
530000000	27850698	27852381	27851033	+1683	+335
540000000	28348381	28349949	28348589	+1568	+208
550000000	28845356	28847059	28845688	+1703	+332
560000000	29342150	29343720	29342338	+1570	+188
570000000	29838286	29839940	29838548	+1654	+262
580000000	30334175	30335730	30334327	+1555	+152
590000000	30829544	30831095	30829682	+1551	+138
600000000	31324703	31326045	31324622	+1342	−81
610000000	31819444	31820587	31819153	+1143	−291
620000000	32313388	32314729	32313285	+1341	−103
630000000	32807229	32808477	32807023	+1248	−206
640000000	33300450	33301838	33300374	+1388	−76
650000000	33793395	33794819	33793345	+1424	−50
660000000	34286170	34287427	34285943	+1257	−227
670000000	34778319	34779667	34778173	+1348	−146
680000000	35270167	35271546	35270042	+1379	−125
690000000	35761747	35763069	35761556	+1322	−191
700000000	36252931	36254242	36252719	+1311	−212
710000000	36743905	36745071	36743539	+1166	−366
720000000	37234048	37235561	37234019	+1513	−29
730000000	37724170	37725717	37724166	+1547	−4
740000000	38213987	38215544	38213984	+1557	−3
750000000	38703181	38705046	38703477	+1865	+296
760000000	39192219	39194230	39192652	+2011	+433
770000000	39680979	39683099	39681512	+2120	+533
780000000	40169476	40171658	40170062	+2182	+586
790000000	40658253	40659911	40658306	+1658	+53
800000000	41146179	41147862	41146248	+1683	+69
810000000	41634187	41635516	41633893	+1329	−294
820000000	42121502	42122877	42121245	+1375	−257
830000000	42608404	42609948	42608308	+1544	−96
840000000	43095410	43096733	43095084	+1323	−326
850000000	43581966	43583236	43581579	+1270	−387
860000000	44067840	44069462	44067796	+1622	−44
870000000	44553888	44555412	44553738	+1524	−150
880000000	45039361	45041091	45039408	+1730	+47
890000000	45524412	45526502	45524811	+2090	+399
900000000	46009215	46011649	46009949	+2434	+734
910000000	46494557	46496534	46494826	+1977	+269
920000000	46979583	46981161	46979445	+1578	−138
930000000	47463433	47465532	47463809	+2099	+376
940000000	47947424	47949652	47947920	+2228	+496
950000000	48431471	48433523	48431783	+2052	+312
960000000	48915316	48917147	48915399	+1831	+83
970000000	49398798	49400527	49398771	+1729	−27
980000000	49881580	49883667	49881903	+2087	+323
990000000	50364709	50366569	50364797	+1860	+88
1000000000	50847534	50849235	50847455	+1701	−79