The Calculation of Certain Dirichlet Series

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1. Introduction. We will be interested here in the computation of a class of Dirichlet series known as $L_0(s)$. They will be defined presently, and include such examples as

\begin{equation}
L_0(n) = 1 - \frac{1}{3^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \cdots
\end{equation}

and

\begin{equation}
L_{-2}(n) = 1 - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} + \cdots.
\end{equation}

These, and some closely related series, arise in several number-theoretic investigations, including the distribution of primes into arithmetic progressions, the class number of binary quadratic forms, and the distribution of Legendre and Jacobi symbols. Our own immediate interest in them stems from their utility in the calculation of certain other number-theoretic constants. These latter include the $h_n$ of references [1] and [2], the $s_i$ of reference [3], the constant 0.48762 of reference [4], and the constant $\frac{1}{2}C$ of reference [5]. The last of these illustrates our point, for when it was first presented by Bateman and Stemmler [6], it was given as 0.76; subsequently, in [5], the improved value 0.761 was presented; but with the aid of a short table of $L_0(s)$, and utilizing a formula analogous to that in [1, eq. (18), p. 323], it is fairly easy to compute

\begin{equation}
\frac{1}{2}C = 0.7608578.
\end{equation}

Similarly, while Ramanujan [11] gave a certain constant as 0.764 \cdots, and G. Pall [12] gave another as 0.64 \cdots, with the aid of short tables of $L_0(s)$ and $L_0(s)$, one may [13], by a trivial computation, obtain

\begin{equation}
b_1 = 0.764223654
\end{equation}

and

\begin{equation}
b_3 = 0.638909405.
\end{equation}

It may be argued that such precision as in (3) is not needed in these investigations. While that is irrelevant for our present paper, it is not inappropriate to mention briefly some counterarguments. Consider, for example, $P_2(N)$, the number of primes of the form $n^2 + 2$ for $1 \leq n \leq N$. The Hardy-Littlewood conjecture [1] claims that

\begin{equation}
P_2(N) \sim 0.35653155 \text{ li}(N),
\end{equation}

where $\text{li}(N)$ is the logarithmic integral. While an examination of the evidence makes it almost certain that $P_2(N)$ is of the order of $\text{li}(N)$, the heuristic argument in favor of (4) is not that convincing that one would be greatly astonished if the true coefficient were found to differ slightly from that conjectured. For empirical

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tests, therefore, fairly accurate coefficients are highly desirable. Secondly, one might (more ambitiously) attempt to investigate the second term, $S(N)$, in the presumed asymptotic series,

$$P_2(N) \sim 0.35653155 \text{li}(N) + S(N),$$

but to do this an accurate subtraction of the leading term is absolutely essential. Such a second-term investigation arises from an assertion of Ramanujan concerning a certain $B(x)$. In effect, he states that

$$B(x) = \frac{b_1 x}{\sqrt{\log x}} \left[ 1 + \frac{1}{2 \log x} + O \left( \frac{1}{\log^2 x} \right) \right].$$

But G. H. Hardy [11, p. 63], and Miss Stanley [14], disagree with the second term here, and go as far as to state (erroneously, [15]) that Ramanujan’s approximation is no better than that of the leading term. Finally, consider a question such as this: Are there more primes (asymptotically speaking) of the form $n^2 + 2$ or of the form $n^2 + 6$? The Hardy-Littlewood conjecture [2] gives

$$\frac{P_3(N)}{P_6(N)} \sim 1.0000301,$$

but it is clear that this very slight excess of the first type could not have been found unless the pertinent constants were available to at least 5 or 6 decimals.

Let $n$ be an odd integer, $\alpha$ an arbitrary integer, and $(\alpha/n)$ the Jacobi symbol. That is, if $n$ is a product of odd primes, $n = \Pi p_i$, then $(\alpha/n) = \Pi (\alpha/p_i)$, where $(\alpha/p_i)$ is the Legendre symbol. If $\alpha$ is not prime to $n$, $(\alpha/n) = 0$.

Now, for every $a$, we define [1, p. 323]

$$L_a(s) = \sum_{\text{odd } n} \left( \frac{-a}{n} \right) \frac{1}{n^s}. \tag{5}$$

The values of $(-a/n)$ are periodic in $n$ and always either $+1$, $-1$, or 0. As examples we list the complete periods of these coefficients of $n^{-s}$ (odd $n$) for twelve of our series.

$$L_1(s) : \quad + -$$
$$L_2(s) : \quad + + - -$$
$$L_3(s) : \quad + 0 -$$

$$L_5(s) : \quad + + 0 + + - - 0 - -$$
$$L_6(s) : \quad + 0 + + 0 + - 0 - 0 -$$
$$L_7(s) : \quad - - 0 + + - -.$$

$$L_{-1}(s) : \quad +$$
$$L_{-2}(s) : \quad + - - +$$
$$L_{-3}(s) : \quad + 0 - - 0 +$$

$$L_{-5}(s) : \quad + - 0 - +$$
$$L_{-6}(s) : \quad + 0 - 0 - 0 - 0 - + 0 +$$
$$L_{-7}(s) : \quad + 0 - + - + - + 0 - + - +.$$
The reader may easily verify that \( L_{±4}(s) = L_{±1}(s) \), and that \( L_{±5}(s) = L_{±3}(s) \), but that

\[
L_{±9}(s) = \left(1 \pm \frac{1}{3^s}\right) L_{±1}(s),
\]

so that, for instance,

\[
\begin{align*}
L_4(s): & \quad + - \\
L_5(s): & \quad + + - - \\
L_9(s): & \quad + 0 + - 0 -.
\end{align*}
\]

While \( L_a(s) \) is an analytic function of the complex variable \( s \) that may be extended to the entire complex plane, our interest in it here will be confined to integral values of \( s \).

Now \( L_i(s) \) and \( L_{±i}(s) \) have been tabulated and are well-known under the more common names: \( L_i(s) = L(s) \) and \( L_{±i}(s) = (1 - 2^{-s}) \zeta(s) \). They need no further treatment here. Some of the other \( L_a(s) \) have also been (partially) investigated long ago. Thus [7], [8] Glaisher’s \( h_n \) is our \( L_a(n) \), his \( q_n \) is our \( L_{±2}(n) \), his \( p_n \) is our \( L_2(n) \), and his \( l_n \) is our \( L_{±3}(n) \). But this unsystematic notation of Glaisher is inadequate here, since we have defined \( L_a(s) \) for infinitely many \( a \). Further, it is clearly desirable for the notation to simply and unequivocally define the series in question. This is accomplished by the notation \( L_a(s) \), as we have seen. (We may similarly criticize, at least for some purposes, the common notation \( L(s, \chi) \). Unless \( \chi \), the particular \textit{character} in question is fully defined, this notation is certainly ambiguous.)

Some of the \( L_a(s) \) may be expressed in closed form. Thus, Glaisher gives

\[
h_{2n+1} = \frac{1}{\sqrt{3}} \left(\frac{\pi}{3}\right)^{2n+1} \frac{H_n}{(2n)!},
\]

where the coefficient \( H_n \) is generated by

\[
3 \cos \left(\frac{1}{2} t\right) = \sum_{n=0}^{\infty} \frac{H_n t^{2n}}{(2n)!}.
\]

Likewise, he gives

\[
q_{2n} = \sqrt{2} \left(\frac{\pi}{4}\right)^{2n} \frac{Q_n}{(2n - 1)!},
\]

where

\[
\frac{\sin t}{\cos(2t)} = \sum_{n=1}^{\infty} \frac{Q_n t^{2n-1}}{(2n - 1)!}.
\]

Here again, these individualized formulas and generating functions do not suffice for our more general needs.

We give below a brief presentation of the general theory of those \( L_a(s) \) that may be expressed in closed form when \( s \) is an integer. This is a generalization and adaptation of the presentation given by Landau [9] for the evaluation of his \( K(d) \).
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As it develops, our treatment is somewhat simpler than Glaisher’s, even for those examples that he did give. Thus [7, p. 64] Glaisher gives a two-term formula:

\[ q_{2n} = \sqrt{2}(-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n-1)!} \left( A_{2n} \left( \frac{1}{8} \right) - \frac{1}{2^{2n}} A_{2n} \left( \frac{1}{4} \right) \right), \]

while we obtain a one-term formula:

\[ L_{-2}(s) = \frac{2}{\sqrt{2}} C_s \left( \frac{1}{8} \right). \]

Similarly, in [8, p. 102], we see a four-term formula:

\[
\sqrt{6} \left( 1 + \frac{1}{5^r} - \frac{1}{7^r} - \frac{1}{11^r} - \frac{1}{13^r} - \frac{1}{17^r} + \frac{1}{19^r} + \frac{1}{23^r} + \cdots \right) \\
= (-1)^{r-1} \left( b_{2r} \left( \frac{\pi}{12} \right) + b_{2r} \left( \frac{5\pi}{12} \right) - b_{2r} \left( \frac{7\pi}{12} \right) - b_{2r} \left( \frac{11\pi}{12} \right) \right)
\]

while we obtain the two-term formula:

\[ L_{-6}(s) = \frac{2}{\sqrt{6}} \left( C_s \left( \frac{1}{24} \right) + C_s \left( \frac{5}{24} \right) \right). \]

This relative simplicity stems from our use of \( C_s(x) \), since, as we shall see, the series for this function, unlike those for the \( A_{2n}(x) \) and \( b_{2r}(x) \) above, involves only the powers of the successive odd integers.

In Sections 2 and 3 we give three tables of closed-form expressions for all the \( L_a(s) \) that may be so expressed, with a range in \( s \) from 1 to 10, and for values of \( a = \pm 2, \pm 3, \pm 5, \pm 6, \pm 7, \pm 10, \pm 11, \pm 13, \pm 14, \) and \( \pm 15 \). In Section 4 we present the theory of \( L_a(s) \) for integral values of \( s < 1 \).

In Section 5 we shall discuss computational techniques for evaluating those \( L_a(s) \) not obtainable in closed form, as, for example, \( L_0(2m) \). In that section we give twelve tables of \( L_a(s) \) to 30D. Here \( s = 1 (1) 10 \), and \( a = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \) and \( \pm 18 \). The first two tables are well-known, but are reproduced here for the reader’s convenience. The last four are rational multiples of the first four, namely,

\[ L_{\pm 9}(s) = \left( 1 \pm \frac{1}{3^s} \right) L_{\pm 1}(s), \]

and

\[ L_{\pm 18}(s) = \left( 1 \pm \frac{1}{3^s} \right) L_{\pm 2}(s). \]

Taken together we have all four characters [9, p. 109] modulo 8:

\[ L_{-3}(s) : \quad + + + + \]
\[ L_1(s) : \quad + + + - \]
\[ L_{-2}(s) : \quad + - - + \]
\[ L_2(s) : \quad + + - - \]
all four characters modulo 12:

\[
\begin{align*}
L_9(s): & \quad +0++0+ \\
L_9(s): & \quad +0++0+ \\
L_{-9}(s): & \quad +0--0+ \\
L_9(s): & \quad +0++0+ \\
\end{align*}
\]

and all eight characters modulo 24:

\[
\begin{align*}
L_9(s): & \quad +0++0++0++0+ \\
L_9(s): & \quad +0++0++0++0+ \\
L_{-9}(s): & \quad +0--0++0++0+ \\
L_{18}(s): & \quad +0++0++0++0+ \\
L_9(s): & \quad +0++0++0++0+ \\
L_{-9}(s): & \quad +0--0++0++0+ \\
L_9(s): & \quad +0++0++0++0+ \\
L_9(s): & \quad +0++0++0++0+ \\
\end{align*}
\]

It follows that the reader may readily compute the following arithmetical-progression Dirichlet series by simple linear combinations:

\[
\begin{align*}
\sum_{k=0}^{\infty} \frac{1}{(8k+1)^s} & = \frac{1}{4}[L_{-1}(s) + L_1(s) + L_{-2}(s) + L_2(s)], \\
\sum_{k=0}^{\infty} \frac{1}{(8k+3)^s} & = \frac{1}{4}[L_{-1}(s) - L_1(s) - L_{-2}(s) + L_2(s)], \\
\sum_{k=0}^{\infty} \frac{1}{(8k+5)^s} & = \frac{1}{4}[L_{-1}(s) + L_1(s) - L_{-2}(s) - L_2(s)], \\
\sum_{k=0}^{\infty} \frac{1}{(8k+7)^s} & = \frac{1}{4}[L_{-1}(s) - L_1(s) + L_{-2}(s) - L_2(s)].
\end{align*}
\]

Similarly, one may obtain

\[
\sum_{k=0}^{\infty} \frac{1}{(12k+b)^s} \quad \text{or} \quad \sum_{k=0}^{\infty} \frac{1}{(24k+b)^s}
\]

for \( b \) prime to 12 or 24, respectively, since, by weighting the character series by the coefficient that appears in the \( c \)th column in the foregoing arrays, one obtains, as the sum, 4, or 8, times the corresponding arithmetical-progression series for \( b = 2c - 1 \).

It was, of course, just such linear combinations, and their utility for his investigation of primes in arithmetical progressions, that led Dirichlet to invent these series.

As for

\[
\sum_{k=0}^{\infty} \frac{1}{(mk+b)^s}
\]

for \( m = 8, 12, \) or 24 and \( b \) not prime to \( m \), those are also obtainable, with slightly more arithmetic. For example,

\[
\sum_{k=0}^{\infty} \frac{1}{(8k+2)^s} = \frac{1}{2^{s+1}} [L_{-1}(s) + L_1(s)].
\]

2. The Theory of the Closed Form Evaluations. The principles behind the closed-form evaluations are first, the trigonometric evaluation of the Jacobi symbol, and
second, the subsequent evaluation of the derived Fourier series. Consider, for example:

\[ \left( \frac{-5}{n} \right) : \quad + + 0 + + + + 0 + + , \text{ repeat.} \]

It is periodic, with a period of 20, and antisymmetric. It is therefore expressible as a (finite) Fourier sine series. In fact, if one evaluates

\[ \sin \left[ (2k + 1) \frac{\pi}{10} \right] + \sin \left[ (2k + 1) \frac{3\pi}{10} \right] \]

for \( k = 0, 1, 2, \ldots \), one obtains the repeating sequence:

\[ + A, + A, 0, + A, + A, - A, - A, 0, - A, - A, \text{ repeat}, \]

where \( A \) is equal to \( \frac{1}{4} \sqrt{5} \). The needed generalization for any \( \left( \frac{\alpha}{n} \right) \) is obtained by the use of Gauss Sums.

Let \( d \) be an integer satisfying the two conditions [9, p. 219]:

(a) \( d \equiv 1 \) (mod 4) or \( d \equiv 8 \) or 12 (mod 16);

(b) \( p^2 \nmid d \) for any odd prime \( p \).

Some of the admissible values of \( d \) are \(-24, -20, -8, -7, -4, -3, 5, 8, 12, 24, 28\). For any admissible \( d \) define the Kronecker symbol \( (d/n) \) for \( n = 1, 2, 3, \ldots \), as follows [9, p. 70–72]. If \( n \) is odd, \( (d/n) \) is the Jacobi symbol; if \( n \) and \( d \) are both even, \( (d/n) = 0 \); and if \( n \) is even, \( d \) odd, then \( (d/n) \) equals the Jacobi symbol \( (n/|d|) \).

For any admissible \( d \) we now have an identity [9, p. 221] that generalizes the Gauss Sum, namely

\[ \left( \frac{d}{n} \right) = \frac{1}{\sqrt{d}} \sum_{r=1}^{\frac{|d|}{2}} \left( \frac{d}{r} \right) e^{2\pi irn/|d|}. \]

Inserting the appropriate Jacobi symbols on the right, the Kronecker symbol on the left is given as a linear combination of complex exponentials, that is, as a linear combination of sines or cosines. Using the abbreviations

\[ s_n(x) = \sin(2\pi nx), \quad c_n(x) = \cos(2\pi nx), \]

the reader may verify the following evaluations of \( (d/n) \) for the 11 values of \( d \) mentioned above.

\[ \left( \frac{-4}{n} \right) = s_n \left( \frac{1}{4} \right) \]
\[ \left( \frac{-8}{n} \right) = \frac{1}{\sqrt{2}} \left[ s_n \left( \frac{1}{8} \right) + s_n \left( \frac{3}{8} \right) \right] \]
\[ \left( \frac{-3}{n} \right) = \frac{2}{\sqrt{3} s_n \left( \frac{1}{3} \right)} \]
\[ \left( \frac{-20}{n} \right) = \frac{1}{\sqrt{5}} \left[ s_n \left( \frac{1}{20} \right) + s_n \left( \frac{3}{20} \right) + s_n \left( \frac{7}{20} \right) + s_n \left( \frac{9}{20} \right) \right] \]
\[ \left( \frac{-24}{n} \right) = \frac{1}{\sqrt{6}} \left[ s_n \left( \frac{1}{24} \right) + s_n \left( \frac{5}{24} \right) + s_n \left( \frac{7}{24} \right) + s_n \left( \frac{11}{24} \right) \right] \]
\[ \left( \frac{-7}{n} \right) = \frac{2}{\sqrt{7}} \left[ s_n \left( \frac{1}{7} \right) + s_n \left( \frac{2}{7} \right) - s_n \left( \frac{3}{7} \right) \right] \]
\begin{align*}
\left( \frac{8}{n} \right) &= \frac{1}{\sqrt{2}} \left[ c_n \left( \frac{1}{8} \right) - c_n \left( \frac{3}{8} \right) \right] \\
\left( \frac{12}{n} \right) &= \frac{1}{\sqrt{3}} \left[ c_n \left( \frac{1}{12} \right) - c_n \left( \frac{5}{12} \right) \right] \\
\left( \frac{5}{n} \right) &= \frac{2}{\sqrt{5}} \left[ c_n \left( \frac{1}{5} \right) - c_n \left( \frac{2}{5} \right) \right] \\
\left( \frac{24}{n} \right) &= \frac{1}{\sqrt{6}} \left[ c_n \left( \frac{1}{24} \right) + c_n \left( \frac{5}{24} \right) - c_n \left( \frac{7}{24} \right) - c_n \left( \frac{11}{24} \right) \right] \\
\left( \frac{28}{n} \right) &= \frac{1}{\sqrt{7}} \left[ c_n \left( \frac{1}{28} \right) + c_n \left( \frac{3}{28} \right) - c_n \left( \frac{5}{28} \right) + c_n \left( \frac{9}{28} \right) - c_n \left( \frac{11}{28} \right) - c_n \left( \frac{13}{28} \right) \right].
\end{align*}

For our \( L_n(s) \) we are only interested in the Jacobi symbol, that is, in such formulas for odd \( n \). But for odd \( n \) we have two simplifications.

(a) On the right, since
\[ c_n \left( \frac{1}{4} \right) = 0, \]
we have
\begin{align*}
\left( \frac{8}{n} \right) &= s_n \left( \frac{1}{4} \right) \\
\left( \frac{12}{n} \right) &= \frac{2}{\sqrt{2}} s_n \left( \frac{1}{8} \right) \\
\left( \frac{5}{n} \right) &= \frac{2}{\sqrt{5}} s_n \left( \frac{1}{5} \right) \\
\left( \frac{24}{n} \right) &= \frac{1}{\sqrt{6}} \left[ s_n \left( \frac{1}{24} \right) + s_n \left( \frac{5}{24} \right) - s_n \left( \frac{7}{24} \right) - s_n \left( \frac{11}{24} \right) \right] \\
\left( \frac{28}{n} \right) &= \frac{1}{\sqrt{7}} \left[ s_n \left( \frac{1}{28} \right) + s_n \left( \frac{3}{28} \right) - s_n \left( \frac{5}{28} \right) + s_n \left( \frac{9}{28} \right) - s_n \left( \frac{11}{28} \right) - s_n \left( \frac{13}{28} \right) \right].
\end{align*}

(13)

(b) On the left, if \( n \) is odd,
\[ s_n \left( \frac{1}{4} + y \right) = s_n \left( \frac{1}{4} - y \right) \\
c_n \left( \frac{1}{4} + y \right) = -c_n \left( \frac{1}{4} - y \right), \]
and some terms may therefore be combined.

(14)

(b) On the left, if \( n \) is odd,
\[ \left( \frac{4k}{n} \right) = \left( \frac{k}{n} \right). \]

Making these simplifications, we thus evaluate the following Jacobi symbols:
\begin{align*}
\left( \frac{1}{n} \right) &= s_n \left( \frac{1}{4} \right) \\
\left( \frac{3}{n} \right) &= \frac{2}{\sqrt{2}} s_n \left( \frac{1}{8} \right) \\
\left( \frac{3}{n} \right) &= \frac{2}{\sqrt{3}} s_n \left( \frac{1}{3} \right) \\
\left( \frac{5}{n} \right) &= \frac{2}{\sqrt{5}} \left[ s_n \left( \frac{1}{20} \right) + s_n \left( \frac{3}{20} \right) \right] \\
\left( \frac{6}{n} \right) &= \frac{2}{\sqrt{6}} \left[ s_n \left( \frac{1}{24} \right) + s_n \left( \frac{5}{24} \right) \right] \\
\left( \frac{7}{n} \right) &= \frac{2}{\sqrt{7}} \left[ s_n \left( \frac{1}{14} \right) + s_n \left( \frac{2}{7} \right) - s_n \left( \frac{3}{7} \right) \right].
\end{align*}

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\[
\begin{align*}
\left( \frac{2}{n} \right) &= \frac{2}{\sqrt{2}} c_n \left( \frac{1}{8} \right), \\
\left( \frac{3}{n} \right) &= \frac{2}{\sqrt{3}} c_n \left( \frac{1}{12} \right), \\
\left( \frac{5}{n} \right) &= \frac{2}{\sqrt{5}} \left[ c_n \left( \frac{1}{5} \right) - c_n \left( \frac{2}{5} \right) \right], \\
\left( \frac{6}{n} \right) &= \frac{2}{\sqrt{6}} \left[ c_n \left( \frac{1}{24} \right) + c_n \left( \frac{5}{24} \right) \right], \\
\left( \frac{7}{n} \right) &= \frac{2}{\sqrt{7}} \left[ c_n \left( \frac{1}{28} \right) + c_n \left( \frac{3}{28} \right) - c_n \left( \frac{5}{28} \right) \right].
\end{align*}
\]

Substitution of these evaluations in (5) converts our Dirichlet series into Fourier series. (It is, of course, not merely a coincidence that the first rigorous work on the latter was done by Dirichlet.) We abbreviate

\[
S_s(x) = \sum_{k=0}^{\infty} \frac{\sin 2\pi (2k+1)x}{(2k + 1)^s}
\]

\[
C_s(x) = \sum_{k=0}^{\infty} \frac{\cos 2\pi (2k+1)x}{(2k + 1)^s}
\]

and obtain

\[
\begin{align*}
L_1(s) &= S_s \left( \frac{1}{4} \right), \\
L_2(s) &= \frac{2}{\sqrt{2}} S_s \left( \frac{1}{8} \right), \\
L_3(s) &= \frac{2}{\sqrt{3}} S_s \left( \frac{1}{3} \right), \\
L_5(s) &= \frac{2}{\sqrt{5}} \left[ S_s \left( \frac{1}{20} \right) + S_s \left( \frac{3}{20} \right) \right], \\
L_6(s) &= \frac{2}{\sqrt{6}} \left[ S_s \left( \frac{1}{24} \right) + S_s \left( \frac{5}{24} \right) \right], \\
L_7(s) &= \frac{2}{\sqrt{7}} \left[ S_s \left( \frac{1}{7} \right) + S_s \left( \frac{2}{7} \right) - S_s \left( \frac{3}{7} \right) \right],
\end{align*}
\]

\[
\begin{align*}
L_{-2}(s) &= \frac{2}{\sqrt{2}} C_s \left( \frac{1}{8} \right), \\
L_{-3}(s) &= \frac{2}{\sqrt{3}} C_s \left( \frac{1}{12} \right), \\
L_{-5}(s) &= \frac{2}{\sqrt{5}} \left[ C_s \left( \frac{1}{5} \right) - C_s \left( \frac{2}{5} \right) \right], \\
L_{-6}(s) &= \frac{2}{\sqrt{6}} \left[ C_s \left( \frac{1}{24} \right) + C_s \left( \frac{5}{24} \right) \right], \\
L_{-7}(s) &= \frac{2}{\sqrt{7}} \left[ C_s \left( \frac{1}{28} \right) + C_s \left( \frac{3}{28} \right) - C_s \left( \frac{5}{28} \right) \right].
\end{align*}
\]
The Fourier series $S_{2m-1}(x)$ and $C_{2m}(x)$ may, in turn, be evaluated in closed form for $m = 1, 2, 3, \cdots$.

Thus,

\begin{align*}
S_1(x) &= \frac{1}{4} \pi \\
C_2(x) &= \frac{1}{2} \pi^2 \left( \frac{1}{4} - x \right) \\
S_3(x) &= \frac{1}{2} \pi^3 \left( \frac{1}{2} x - x^2 \right) \\
C_4(x) &= \pi^4 \left( \frac{1}{96} - \frac{1}{4} x^2 + \frac{1}{3} x^3 \right) \\
S_5(x) &= \frac{1}{6} \pi^5 \left( \frac{1}{8} x - x^3 + x^4 \right) \\
C_6(x) &= \frac{1}{3} \pi^6 \left( \frac{1}{320} - \frac{1}{16} x^2 + \frac{1}{4} x^4 - \frac{1}{5} x^5 \right) \\
S_7(x) &= \frac{1}{3} \pi^7 \left( \frac{1}{160} x - \frac{1}{24} x^3 + \frac{1}{10} x^5 - \frac{1}{15} x^6 \right) \\
C_8(x) &= \frac{2}{3} \pi^8 \left( \frac{17}{105 \cdot 1024} x^2 + \frac{1}{96} x^4 - \frac{1}{60} x^6 + \frac{1}{105} x^7 \right) \\
S_9(x) &= \frac{1}{45} \pi^9 \left( \frac{17}{7 \cdot 256} x - \frac{1}{10} x^3 + \frac{1}{8} x^5 - \frac{1}{7} x^7 + \frac{1}{14} x^8 \right) \\
C_{10}(x) &= \frac{1}{45} \pi^{10} \left( \frac{31}{63 \cdot 1024} - \frac{17}{7 \cdot 256} x^2 + \frac{1}{32} x^4 - \frac{1}{24} x^6 + \frac{1}{28} x^8 - \frac{1}{63} x^9 \right).
\end{align*}

These formulas may be verified by the relations

\begin{align*}
S_n(x) &= -\frac{1}{2\pi} \frac{d}{dx} C_{n+1}(x), \\
C_n(x) &= \frac{1}{2\pi} \frac{d}{dx} S_{n+1}(x),
\end{align*}

and

\[ C_{2n}(0) = -C_{2n}(\frac{1}{2}). \]

The latter relation fixes the constant term in $C_{2n}(x)$. We may note that $C_{2n}(0) = L_{-1}(2n)$. The formulas may clearly be extended to larger indices by integration.

One may therefore obtain, in closed form, $L_a(2n + 1)$ for positive $a$ and $L_{-a}(2n)$ for positive $a$. Let us define $C_{a,n}$ and $D_{a,n}$ for positive $a$ by the equations

\begin{align*}
L_a(2n + 1) &= \left( \frac{x}{a} \right)^{2n+1} \sqrt{a} C_{a,n} \\
L_{-a}(2n) &= \left( \frac{x}{a} \right)^{2n} \sqrt{a} D_{a,n}.
\end{align*}
We will give presently tables of such $C_{a,n}$ and $D_{a,n}$ up to $n = 4$ and 5, respectively.

To generalize the results of (19), we proceed as follows. Let $a = N^2b$, where $b$ is not divisible by a square $> 1$. Henceforth, the letter $b$ here refers to such a number. Then it is readily seen that

$$(23) \quad L_a(s) = \prod_{p} \left[ 1 - \left( \frac{-b}{p_1} \right) \frac{1}{p_i^s} \right] L_b(s),$$

the product on the right being taken over each odd prime $p_i$ that divides $N$. Now if

$$-b \equiv 1 \pmod{4}$$

we evaluate the Kronecker symbol $(-b/n)$ by (10), while if

$$-b \equiv 2, 3 \pmod{4},$$

we evaluate $(-4b/n)$ by (10). The corresponding Jacobi symbol $(-b/n)$ is now inserted into $L_b(s)$ as before. We must distinguish four cases in our final result.

I. For $b > 0$, $b \equiv 3 \pmod{4}$:

$$L_b(s) = \frac{2}{\sqrt{b}} \sum_{k=1}^{(b-1)/2} \left( \frac{k}{b} \right) S_s(k/b).$$

II. For $b < 0$, $-b \equiv 1 \pmod{4}$:

$$L_b(s) = \frac{2}{\sqrt{-b}} \sum_{k=1}^{(-b-1)/2} \left( \frac{k}{-b} \right) C_s(k/-b).$$

III. For $b > 0$, $b \not\equiv 3 \pmod{4}$:

$$L_b(s) = \frac{2}{\sqrt{b}} \sum_{\text{odd } k < b} \left( \frac{-b}{k} \right) S_s(k/4b).$$

IV. For $b < 0$, $-b \not\equiv 1 \pmod{4}$:

$$L_b(s) = \frac{2}{\sqrt{-b}} \sum_{\text{odd } k < b} \left( \frac{-b}{k} \right) C_s(k/-4b).$$

Using these and (20) and (22), we thus compute Tables 1 and 2.

3. Another Class of Closed Forms. There is another class of closed form evaluations, $L_{-b}(1)$ for $b > 0$. This is well known. We use the Fourier series

$$(28) \quad C_1(x) = \frac{1}{2} \log \cot \pi x,$$

and from (25) and (27) we thus obtain for

$$b > 0, \ b \equiv 1 \pmod{4}$$

$$(29) \quad L_{-b}(1) = \frac{1}{\sqrt{b}} \log \prod_{k=1}^{(b-1)/2} \left( \cot \left( \frac{\pi k}{b} \right) \right)^{\left( \frac{k}{b} \right)}$$

and for

$$b > 0, \ b \not\equiv 1 \pmod{4}$$

$$(30) \quad L_{-b}(1) = \frac{1}{\sqrt{b}} \log \prod_{\text{odd } k < b} \left( \cot \left( \frac{\pi k}{4b} \right) \right)^{\left( \frac{k}{b} \right)}.$$
Table 1. Values of $C_{a,n}$

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<th>$a$</th>
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For large values of $b$ these formulas are not well adapted for easy and accurate computation. To supplement them, we now could follow the $K(d)$ theory in Landau [9], but the earlier pre-Kronecker theory of Gauss and Dirichlet is more direct for our purpose. (We register, in this connection, an objection to the frequent remark that Gauss was not wise in his choice of

$$f = Ax^2 + 2Bxy + Cy^2$$

(31)

as the basis for his theory of binary quadratic forms, since for some purposes, such as our present one, and likewise for investigations in the distribution of Jacobi symbols, the Gauss theory is really simpler and more direct.) An exposition of this earlier theory in English was given by Mathews [10, Chapter VIII].

In [10, p. 238] we have directly

$$L_\alpha(1) = \frac{H(b) \log (T + U\sqrt{b})}{2\sqrt{b}}$$

(32)
Table 2. Values of $D_{a,n}$

<table>
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<tr>
<th>$a$</th>
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where $T + U\sqrt{b}$ is the smallest number $> 1$ such that

$T^2 - bU^2 = 1$,

and $H(b)$ is the number of classes of properly primitive forms (31) for which $D = b$. Mathews also gives formulas similar to our (29) and (30), but slightly more complicated [10, p. 251-252].

The $T$ and $U$ of (33) are easily computed, and if one can obtain the integer $H(b)$, (32) would suffice. But if $H(b)$ is not available, one can proceed by the following combined operation. Estimate the logarithm on the right side of (29) or (30) with sufficient accuracy to identify the integer $H(b)$ by

$H(b) = 2 \log(T + U\sqrt{b})$.

Then compute $L_{a,b}(1)$ accurately by (32). In Table 3, equation (32) is written as

$L_{a,b}(1) = \frac{\log(t + u\sqrt{b})}{\sqrt{b}}$,
Table 3

<table>
<thead>
<tr>
<th>b</th>
<th>$t + u \sqrt{b}$</th>
<th>b</th>
<th>$t + u \sqrt{b}$</th>
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<td>$8 + 3\sqrt{7}$</td>
<td>15</td>
<td>$31 + 8\sqrt{15}$</td>
</tr>
</tbody>
</table>

where

$$t + u\sqrt{b} = (T + U\sqrt{b})^{H(b)/2}.$$  

If $H(b)$ is odd, say for $b = 2, 5$, then

$$t^2 - bu^2 = -1;$$

otherwise

$$t^2 - bu^2 = 1.$$  

One can, of course, compute $t + u\sqrt{b}$ by (29) or (30). An interesting and useful transformation of (29) is given when $b > 0$, $b \equiv 1 \pmod{4}$ by

$$(29a) \quad t + u\sqrt{b} = \frac{2\varphi(b/2)}{F(b)} \prod_{k=1}^{(b-1)/2} \left\{ 1 + \left( \frac{k}{b} \right) \cos \frac{2\pi k}{b} \right\}.$$  

Here $\varphi(b)$ is Euler's function, and $F(b) = \sqrt{b}$ or 1, according as $b$ is prime or not. Similarly, when $b > 0$, $b \not\equiv 1 \pmod{4}$,

$$(30a) \quad t + u\sqrt{b} = \frac{2\varphi(b/4)}{ \prod_{\text{odd } k < b} \left\{ 1 + \left( \frac{b}{k} \right) \cos \frac{\pi k}{2b} \right\} }.$$  

For example, for $b = 15$,

$$31 + 8\sqrt{15} = 16 (1 + \cos 6°)(1 + \cos 42°)(1 + \cos 66°)(1 - \cos 78°).$$

4. Nonpositive Arguments. To compute $L_n(s)$ for $s$ an integer $< 1$ we need an analytic continuation of the series in (5). In [16] Landau gives, in effect, a functional equation for certain Dirichlet series $L(s, \chi)$. If $\chi$ is a real, "eigenlichter" character modulo $k > 2$, we have

$$L(1 - s, \chi) = L(s, \chi) \left( \frac{k}{\pi} \right)^{s-1/2} \frac{\Gamma \left( \frac{s + \alpha}{2} \right)}{\Gamma \left( \frac{1 - s + \alpha}{2} \right)}$$

where $\alpha = 0$ if $\chi(-1) = 1$ and $\alpha = 1$ is $\chi(-1) = -1$. In the first case we therefore have

$$L(1 - s, \chi) = L(s, \chi) \frac{2}{\sqrt{k}} \Gamma(s) \left( \frac{k}{2\pi} \right)^s \cos \frac{\pi s}{2}.$$
and in the second,
\begin{equation}
L(1 - s, \chi) = L(s, \chi) \frac{2}{\sqrt{k}} \Gamma(s) \left( \frac{k}{2\pi} \right)^s \sin \frac{\pi s}{2}.
\end{equation}

For the \( L_b(s) \) in our Case III, given by equation (26), equation (39) is valid with \( k = 4b \); and in our Case IV, given by equation (27), equation (38) is valid with \( k = -4b \). For our Cases I and II we must make our \( L_b(s) \) depend upon appropriate related functions. In Case I, \( b = 4m + 3 \), since
\[
L_b(s) = \sum_{n \geq 0} \left( -\frac{4m - 3}{n} \right) \frac{1}{n^s}
\]
we utilize
\begin{equation}
L_b(s) = \left[ 1 - \left( \frac{2}{4m + 3} \right) \left( \frac{1}{2^s} \right) \sum_{n=1}^{\infty} \left( \frac{n}{4m + 3} \right) \frac{1}{n^s} \right].
\end{equation}

For example,
\[
L_b(s) = \left( 1 + \frac{1}{2^s} \right) \sum_{n=1}^{\infty} \left( \frac{n}{3} \right) \frac{1}{n^s}
\]
and
\[
L_b(s) = \left( 1 - \frac{1}{2^s} \right) \sum_{n=1}^{\infty} \left( \frac{n}{3} \right) \frac{1}{n^s}.
\]
In (40), the sum on the right, call it \( l_b(s) \), satisfies (39) with \( k = b \). Similarly, in Case II, \( b = -(4m + 1) \),
\begin{equation}
L_b(s) = \left[ 1 - \left( \frac{2}{4m + 1} \right) \left( \frac{1}{2^s} \right) \sum_{n=1}^{\infty} \left( \frac{n}{4m + 1} \right) \frac{1}{n^s} \right],
\end{equation}
and now \( l_b(s) \) satisfies (38) with \( k = -b \).

Thus, it follows that
\[
L_b(0) = 0
\]
in Cases II and IV, and also in Case I if \( b = 8m + 7 \). More generally,
\[
L_b(-2n) = 0 \quad (n \geq 0)
\]
in Cases II and IV. Likewise,
\[
L_b(1 - 2n) = 0 \quad (n \geq 1)
\]
in Cases I and III.

For the remaining nonpositive arguments, we utilize (22) and the foregoing, and obtain
Case I
\begin{equation}
L_b(-2n) = (-1)^n (2n)! \frac{2 - \left( \frac{2}{b} \right) \left( \frac{2}{2^{2n+1}} \right)}{\frac{2}{2^{2n+1}} - \left( \frac{2}{b} \right)} C_{b,n}.
\end{equation}
Case II

\[ L_b(1 - 2n) = (-1)^n(2n - 1)! \frac{2 - \left( \frac{2}{-b} \right)^{2n}}{2^{2n} - \left( \frac{2}{-b} \right)^{2n}} D_{b,n}. \]

Case III

\[ L_b(-2n) = (-1)^n(2n)! 2^{2n+1} C_{b,n}. \]

Case IV

\[ L_b(1 - 2n) = (-1)^n(2n - 1)! 2^{2n} D_{b,n}. \]

Therefore, \( L_b(s) \) is rational for all integers \( s < 1 \). For example, in Case I, \( b = 3 \), we have \( L(0) = \frac{3}{4}, L(-1) = 0, L(-2) = -\frac{1}{3}, L(-3) = 0, L(-4) = \frac{3}{4} \), etc.

5. Numerical Evaluation of the Dirichlet Series. In Tables 4–15 we present 30D values of \( L_{±a}(s) \) for \( a = 1, 2, 3, 6, 9, 18 \) and \( s = 1(1)10 \). The values of \( L_{±1}(s) \) have been included for ease of reference; they have been abridged from 50D values given by Liénard [17], who designated them as \( u_n \) and \( U_n \), respectively.

From these data the entries in Tables 12 and 13 were derived by use of the relation

\[ L_{±1}(s) = (1 ± 3^{-s}) L_{±1}(s), \]

which we have already noted.

Formulas (22) in conjunction with the appropriate coefficients presented in Tables 1 and 2 were used to evaluate \( L_a(2n + 1) \) and \( L_{-a}(2n) \) to at least 35D when \( a = 2, 3, \) and 6. The requisite decimal approximations to powers of \( \pi \) were obtained from a manuscript table [18] of the second author.

Numerical values of \( L_{-a}(1) \) were computed from equation (35) in conjunction with the corresponding entries in Table 3.

Evaluation of \( L_a(2n) \) and \( L_{-a}(2n + 1) \) was accomplished numerically by means of the following relations:

\[ L_a(s) + L_{-a}(s) + L_{+a}(s) + L_{-1}(s) = 4 \sum_{k=0}^{\infty} (8k + 1)^{-s}, \]

\[ L_a(s) + L_{-a}(s) + L_{+a}(s) + L_{-1}(s) = 4 \sum_{k=0}^{\infty} (12k + 1)^{-s}, \]

\[ L_a(s) + L_{-a}(s) + L_{+18}(s) + L_{-18}(s) = 4 \sum_{k=0}^{\infty} (-1)^k (12k + 1)^{-s}, \]

where \( L_{±18}(s) = (1 ± 3^{-s}) L_{±18}(s) \), as noted previously.

The right members of equations (46) and (47) were computed to 37D by means of the series

\[ \sum_{k=0}^{\infty} (nk + 1)^{-s} = 1 + (n + 1)^{-s} + (2n + 1)^{-s} \]

\[ + \sum_{r=0}^{n-1} (-1)^r \left( \frac{8 + r - 1}{8} \right) S_{r+1, n^{-s-r}}, \]
### Table 4. Values of $L_1(s)$

<table>
<thead>
<tr>
<th>$s$</th>
<th>0. 785398163397448309615660845820</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.915965594177219015054603514932</td>
</tr>
<tr>
<td>2</td>
<td>0.968946146259369380483634845847</td>
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<tr>
<td>3</td>
<td>0.988944551741105336108422633228</td>
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<tr>
<td>4</td>
<td>0.996157828077088064006319368631</td>
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<tr>
<td>5</td>
<td>0.99868522218438354438135887860</td>
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<td>6</td>
<td>0.999554507890539909496346549899</td>
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<tr>
<td>7</td>
<td>0.9999826196877405540729958</td>
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### Table 5. Values of $L_{-1}(s)$

<table>
<thead>
<tr>
<th>$s$</th>
<th>1. 233700550136169827354311374985</th>
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<tr>
<td>1</td>
<td>0.51799790246444999724770891323</td>
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<td>2</td>
<td>0.01467803164192054546255465507</td>
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<td>0.004523762795139616135310315005</td>
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<td>0.001447076640942121906478587138</td>
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<td>0.000471548652376554755111631492</td>
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<td>0.000155179025296119302987249296</td>
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<td>0.000051345183843772592817900543</td>
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<td>0.000017041362196877405540729958</td>
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### Table 6. Values of $L_2(s)$

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<tr>
<th>$s$</th>
<th>1. 110720734539591561753970247515</th>
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<td>1</td>
<td>0.064734171043503370392827451462</td>
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<td>0.027722585936858567879256618002</td>
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<td>0.01050894057394752987898207302</td>
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<td>0.003755686395655010984999754471</td>
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<td>0.001301442454434571963006355517</td>
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<td>0.000443474605655003573027505364</td>
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<td>0.00005271377482728194663055508</td>
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<td>0.000016829464626378164567318164</td>
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### Table 7. Values of $L_{-2}(s)$

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<tr>
<th>$s$</th>
<th>0. 623225240140230513394020080251</th>
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<tr>
<td>1</td>
<td>0.872358024954859491769695117021</td>
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<td>0.958384045463094562051669402862</td>
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<td>0.995633856312967456342217717128</td>
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Table 8. Values of $L_0(s)$

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Table 9. Values of $L_{-3}(s)$

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Table 10. Values of $L_6(s)$

<table>
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<th>$98301$</th>
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Table 11. Values of $L_{-6}(s)$

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### Table 12. Values of $L_9(s)$

<table>
<thead>
<tr>
<th>$s$</th>
<th>1. 04719 75511 96597 74615 42144 61093</th>
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<tr>
<td>2</td>
<td>0.01773 95490 85798 90561 62261 27703</td>
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<tr>
<td>3</td>
<td>0.00483 30405 65271 95013 11768 77175</td>
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<td>4</td>
<td>0.00115 37437 37909 10569 00080 97836</td>
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<td>5</td>
<td>0.00025 72430 07446 45110 09955 80025</td>
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<td>6</td>
<td>0.00005 51607 94869 46347 37566 18485</td>
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<td>7</td>
<td>0.00001 15515 61271 75216 18684 27608</td>
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<td>0.00000 23831 73250 45037 19548 91440</td>
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<td>0.00000 04868 94337 25794 05389 45380</td>
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<td>0.00000 00988 28886 61299 59386 29004</td>
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### Table 13. Values of $L_{-9}(s)$

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<tr>
<th>$s$</th>
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<tbody>
<tr>
<td>1</td>
<td>0.99662 27112 32150 95764 82767 77764</td>
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<td>2</td>
<td>0.01284 42424 77056 55299 05201 17370</td>
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<td>3</td>
<td>0.00215 11423 25127 95108 74108 30131</td>
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<td>4</td>
<td>0.00038 99201 49892 12800 12736 47042</td>
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<td>0.00007 33495 12490 89814 52900 01970</td>
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<td>0.00001 40856 67167 42052 79716 62753</td>
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<td>8</td>
<td>0.00000 27395 83286 47197 49422 88581</td>
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<td>9</td>
<td>0.00000 05373 11812 89092 98298 99836</td>
</tr>
<tr>
<td>10</td>
<td>0.00000 01059 86639 41566 20143 11216</td>
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</table>

### Table 14. Values of $L_{18}(s)$

<table>
<thead>
<tr>
<th>$s$</th>
<th>0. 74048 04896 93061 04116 93134 98343</th>
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<tbody>
<tr>
<td>2</td>
<td>0. 94643 03742 60891 88479 36244 01299</td>
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<tr>
<td>3</td>
<td>0. 98965 87864 57715 65795 78026 69187</td>
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<td>4</td>
<td>0. 99803 35215 54511 36097 57019 33138</td>
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<tr>
<td>5</td>
<td>0. 99962 50045 58635 85456 11931 71119</td>
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<td>6</td>
<td>0. 99992 79150 98529 48847 60886 51326</td>
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<td>7</td>
<td>0. 99998 60244 57229 92126 68669 98061</td>
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<td>0. 99999 72699 38252 82646 78138 02628</td>
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<td>0. 99999 94635 60006 86157 22886 88640</td>
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<td>0. 99999 98940 91809 48666 12706 56623</td>
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### Table 15. Values of $L_{-18}(s)$

<table>
<thead>
<tr>
<th>$s$</th>
<th>0. 83096 69868 53640 68452 53601 07001</th>
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<tbody>
<tr>
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<td>0. 96928 66943 94288 82418 85501 30024</td>
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<td>0. 99387 60269 54320 28657 21016 2968</td>
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<td>0. 99872 24021 84019 52246 82201 01612</td>
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<td>0. 99994 37579 23288 47709 09430 40820</td>
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<td>0. 99998 83487 45529 84164 62256 31933</td>
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<td>0. 99999 76076 77896 41674 40568 41145</td>
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<td>0. 99999 95122 88636 78117 26975 47440</td>
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<td>0. 99999 99100 94690 91294 73632 68439</td>
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</table>
where $S''_t = \zeta(t) - 1 - 2^{-t}$. Thus, the required values of $S''_t$ were obtained from tables of $\zeta(t)$, or $S_t$, given by Liénard [17].

The right member of equation (48) was expeditiously evaluated by means of the relation

$$\sum_{k=0}^{\infty} (-1)^k (12k + 1)^{-s} = 2 \sum_{k=0}^{\infty} (24k + 1)^{-s} - \sum_{k=0}^{\infty} (12k + 1)^{-s}.$$

A partial check on the accuracy of these tables was made by use of the relations

$$L_{-2}(s) + L_{-3}(s) - L_2(s) - L_3(s) = 4 \sum_{k=1}^{\infty} (8k - 1)^{-s}$$

and

$$L_{-3}(s) + L_{-6}(s) - L_2(s) - L_6(s) = 4 \sum_{k=1}^{\infty} (12k - 1)^{-s}$$

$$L_{-3}(s) + L_{-6}(s) + L_{-9}(s) + L_{-18}(s) - L_2(s) - L_6(s) - L_9(s) - L_{18}(s)$$

$$= 8 \sum_{k=1}^{\infty} (24k - 1)^{-s}.$$

These check relations are especially easy to apply because the right members involve the same individual terms (except for sign) as the right members of equations (46) and (47) and the series $\sum (24k + 1)^{-s}$, when evaluated by equation (49).

These check relations were satisfied to within one unit in the thirty-third decimal place when the corresponding data on the work sheets were successively substituted therein.

Applied Mathematics Laboratory
David Taylor Model Basin
Washington 7, D.C.

15. Daniel Shanks, "The second-order term in the asymptotic expansion of $B(\alpha)$," (to appear)
17. R. Liénard, Tables fondamentales à 50 décimales des Sommes $S_n, u_n, \sum_s$, Centre de Documentation universitaire, Paris, 1948.