then D_n is also congruent to a diagonal matrix of the stated form. From the law of inertia ([1], p. 296–298) D_n also has p positive and q negative elements and the proof is complete.

From this result Theorem 1 follows as a corollary since, if $w(x)$ is nonnegative, C_n is positive definite and therefore congruent to a diagonal matrix with n positive elements.

As a simple example consider a 2-point quadrature formula of the form

$$\int_{-1}^{1} (3 - 5 \mid x \mid) f(x)\, dx \simeq A_1 f(x_1) + A_2 f(x_2). \tag{3}$$

For this weight function the monomial integrals are $c_0 = 1$, $c_1 = 0$, $c_2 = -1/2$, $c_3 = 0$. There are no real values of x_1, x_2 for which (3) can be made exact for $f(x) = 1, x, x^2, x^3$. There are, however, an infinity of such formulas with real x_1, x_2 which are exact for $f(x) = 1, x, x^2$ and Theorem 2 still applies. One such formula is

$$x_1 = \tfrac{1}{2} \qquad x_2 = 1$$
$$A_1 = 2 \qquad A_2 = -1.$$

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A Partition Problem

By M. H. McAndrew

1. Introduction. The following theorem is proved: *Given integers a, b, c, d, each* $\geqq 2$, *then either there exist integers m, n with* $| m - n | \leqq 1$, *a partition of a into m parts of which each part is coprime to b, and a partition of c into n parts, each part coprime to d; or the same conclusion holds with the roles of a and b reversed and the roles of c and d reversed.*

This question arises in the investigation of the minimum length of input strings required to distinguish two partial automata. Elgot and Rutledge [1] deduce an upper bound for the length of such strings and by using the theorem quoted above show that this upper bound can be attained. In Section 4 we demonstrate by an example that the restriction "$a, b, c, d \geqq 2$" cannot be relaxed.

2. Preliminary Lemmas. In the sequel, all variables are to be taken as strictly positive integers.

Lemma 1. *If* $l > 1$, $l = \prod_{i=1}^{r} P_i$ *where the* p_i *are distinct primes, and if m is even, then there is an a such that*

$$(a, l) = 1,$$
$$(m - a, l) = 1.$$

Received November 16, 1962.

Proof. For each i the congruences

$$(1) \qquad \begin{cases} a_i \not\equiv 0 \pmod{p_i}, \\ a_i \not\equiv m \pmod{p_i} \end{cases}$$

have a common solution. For if $p_i = 2$ then $a_i = 1$ is a solution; otherwise, $p_i \geqq 3$ and equations (1) eliminate at most 2 of the p_i congruence classes mod p_i.

Let $a \equiv a_i \pmod{p_i}$; by the Chinese Remainder Theorem such an a exists (unique mod l). Clearly a satisfies the conclusions of the lemma.

LEMMA 2. *If p is a prime $\geqq 5$, then for any a, m, l, with $(l, p) = 1$, there is a b such that*

$$(2) \qquad b \equiv a \pmod{l},$$

$$(3) \qquad 1 \leqq b \leqq 3lp/5,$$

$$(4) \qquad (b, p) = (m - b, p) = 1.$$

Proof. Let $x_r = a + rl$. At least $p - 2$ of the integers $x_1, x_2, \cdots, x_p$ satisfy

$$x_r \not\equiv 0 \pmod{p}$$

and

$$x_r \not\equiv m \pmod{p}.$$

Let $y_1, y_2, \cdots, y_{p-2}$ be the least positive residues (mod lp) of any $p - 2$ of such solutions. Let $b = \operatorname{Min}(y_i)$. Then clearly equations (2), (4) are satisfied and

$$1 \leqq b \leqq 3l,$$

$$\leqq 3lp/5, \quad \text{since } p \geqq 5.$$

LEMMA 3. *If $t = \prod_{i=1}^{s} p_i$ where the p_i are distinct primes with $p_s \geqq 5$, and if m is even, then there is a b such that*

$$(b, t) = 1,$$

$$(m - b, t) = 1,$$

and

$$1 \leqq b \leqq 3t/5.$$

Proof. If $s = 1$, the result follows from Lemma 2 with $a = l = 1$. If $s > 1$, the result follows from Lemmas 1 and 2 with

$$l = \prod_{i=1}^{s-1} p_i .$$

Definition. We define "a is $P_r(b)$" to mean "There exist $a_1, a_2, \cdots, a_r$ such that $a = \sum_{i=1}^{r} a_i$ and $(a_i, b) = 1 \quad (i = 1, \cdots, r)$."

LEMMA 4. *If m is an even integer $\geqq 6$ and $m \geqq 3n/5$ then m is $P_2(n)$.*

Proof. Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ where the p_i are distinct primes. Let $t = \prod_{i=1}^{r} p_i$ and suppose, without loss of generality, $p_r = \operatorname{Max}(p_i)$. We consider two cases.

(a) $P_r \geqq 5$. Let b be defined as in Lemma 3. Then

$$\begin{aligned} 1 \leqq b &\leqq 3t/5 \\ &\leqq 3n/5, \text{ by definition of } t \\ &\leqq m, \quad \text{by hypothesis} \\ &< m, \quad \text{since } (m - b, n) = 1. \end{aligned}$$

Therefore, $c = m - b$ is a positive integer; i.e., since $m = b + c$, m is $P_2(n)$.

(b) $P_r < 5$. Now $t = 1, 2, 3$ or 6. It may be verified that at least one of the partitions $m = 1 + (m - 1)$ or $m = 5 + (m - 5)$ must have both parts prime to t. The result now follows.

LEMMA 5. *If* $m \geqq 2$, $n \geqq 1$ *and*

either (a) $m \geqq 3n/5 + 1$ *and* $(m, n) \neq (5, 6)$,
or (b) $m \geqq (3n + 8)/5$,
or (c) $m \geqq n$,

then m is either $P_2(n)$ *or* $P_3(n)$.

Proof. (a) If m is an even integer $\geqq 6$, then m is $P_2(n)$ by Lemma 4. If m is an odd integer $\geqq 7$, then $m - 1$ is an even integer $\geqq 3n/5$ and hence $m - 1$ is $P_2(n)$ by Lemma 4; i.e., m is $P_3(n)$. The cases $m = 2, 3, 4, 5$ may be settled by inspection of the following partitions:

$$\begin{aligned} 2 &= 1 + 1, \\ 3 &= 1 + 1 + 1, \\ 4 &= 2 + 2 = 3 - 1 \qquad (n \leqq 5/3(3) = 5), \\ 5 &= 3 + 1 + 1 = 2 + 2 + 1 \qquad (n \leqq 5/3(4) < 7). \end{aligned}$$

(b) If $(m, n) \neq (5, 6)$ the result follows from (a). If $(m, n) = (5, 6)$ then $m < (3n + 8)/5$.

(c) If $n \geqq 3$ then $m \geqq n > 3n/5 + 1$ and the result follows from (a). If $n = 1$ or 2, the result follows since $m = (m - 1) + 1 = (m - 2) + 1 + 1$ and either $m - 1$ or $m - 2$ is odd.

LEMMA 6. *If* $u \geqq w \geqq 2$ *and* $u \geqq w + (3v - 2)/5$ *then u is either* $P_w(v)$ *or* $P_{w+1}(v)$.

Proof.

$$u = \underbrace{1 + 1 + \cdots 1}_{w - 2} + (u - w + 2)$$

and $(u - w + 2) \geqq (3v + 8)/5$, by hypothesis. Hence, by Lemma 5(b), $(u - w + 2)$ is either $P_2(v)$ or $P_3(v)$. Therefore, u is either $P_w(v)$ or $P_{w+1}(v)$.

3. Main Theorem.

THEOREM 1. *If a, b, c, d satisfy*

$$(5) \qquad a, b, c, d, \geqq 2$$

then for some m, n with $|m - n| \leqq 1$, *either a is* $P_m(b)$ *and c is* $P_n(d)$, *or b is* $P_m(a)$ *and d is* $P_n(c)$.

Proof. Suppose the theorem is false. We shall deduce a contradiction. We consider two cases.

(i) Either (a, b), (b, a), (c, d) or $(d, c) = (5, 6)$. Suppose, without loss of generality, $(a, b) = (6, 5)$. Now $6 = 4 + 2$; hence 6 is $P_2(5)$. Therefore c is not $P_2(d)$ or $P_3(d)$. By the converse of Lemma 5 (c), $d > c$; i.e.,

$$d \geqq c + 1. \tag{6}$$

Now $6 = 3 + 3 = 3 + 2 + 1 = 3 + 1 + 1 + 1 = 2 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1$; i.e., 6 is $P_2(5)$, $P_3(5)$, $P_4(5)$, $P_5(5)$, $P_6(5)$. Now c is $P_c(d)$, trivially. Therefore

$$c \geqq 8. \tag{7}$$

Finally 5 is $P_5(6)$, trivially; hence d is not $P_4(c)$ or $P_5(c)$. Therefore, by the converse of Lemma 6 with $u = d$, $v = c$, and $w = 4$,

$$d < (3c + 18)/5. \tag{8}$$

From (6) and (8),

$$c + 1 < (3c + 18)/5, \qquad 2c < 13,$$

which contradicts (7).

(ii) None of (a, b), (b, a), (c, d), $(d, c) = (5,6)$. Without loss of generality, suppose $a \geqq b$. Then by Lemma 5(c) a is either $P_2(b)$ or $P_3(b)$. Hence c is neither $P_2(d)$ nor $P_3(d)$; by the converse of Lemma 5,

$$c < d, \tag{9}$$

$$c < 3d/5 + 1. \tag{10}$$

Similarly from (9) we deduce

$$b < a, \tag{11}$$

$$b < 3a/5 + 1. \tag{12}$$

Suppose without loss of generality, $a \geqq d$. In (10)

$$c < 3a/5 + 1. \tag{13}$$

From (12) and (13)

$$\begin{aligned} 3b + 5c &< 3(3a/5 + 1) + 5(3a/5 + 1) \\ &< \tfrac{24}{5}a + 8 \\ &< 5a + 8 \\ &\leqq 5a + 7. \end{aligned}$$

i.e. $a \geqq (c - 1) + (3b - 2)/5$. Hence, by Lemma 6, a is either $P_{c-1}(b)$ or $P_c(b)$. Now c is $P_c(d)$ trivially; hence we have the required contradiction.

4. Remark. The following theorem shows that Theorem 1 is best possible in that condition (5) cannot be relaxed.

THEOREM 2. *For arbitrary K there exist a, b, c, d, with*

$$a = 1,$$

$$b, c, d, > K$$

and such that the conclusion of Theorem 1 *is false.*

Proof. Let $b = N$, $d = 2^M - 2$, $c = 2^{-r}(d!)$, where r is chosen to make c an odd integer. Clearly a is $P_s(b)$ only for $s = 1$. Now c is not $P_1(d)$, provided $d > 3$, and not $P_2(d)$ since an odd integer cannot be the sum of two odd integers. Hence, we cannot find partitions of a, c satisfying the conclusions of Theorem 1.

Suppose d is $P_s(c)$. Then

$$d = d_1 + d_2 + \cdots + d_s$$

where each $d_i \leqq d$ and $(d_i, c) = 1$. Now c is divisible by all odd integers $< d$; therefore d_i is a power of 2. I.e.,

$$d = 2^{r_1} + 2^{r_2} + \cdots 2^{r_s}. \tag{14}$$

Since $d = 2^M - 2$ there are at least $M - 1$ summands in (14). I.e., if $d = P_s(c)$, then $s \geqq M - 1$. But clearly if b is $P_s(a)$, then $s \leqq N$. If we now choose $M - 1 > N + 1$ and M, N large enough to ensure $b, c, d > K$, the conclusion of Theorem 2 follows.

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Approximations to Kelvin Functions

By F. D. Burgoyne

While preparing a digital computer program to examine the behavior of large-taper hub flanges, it was found necessary to use approximations to the Kelvin functions ber x, bei x, ker x, and kei x, and to their first derivatives. To obtain full machine accuracy, the approximations were required to be correct to nine significant figures. Several tabulations of these functions exist, but the only ones considered to be sufficiently accurate were those of Lowell [1] and Nosova [2]; however, limitations of internal memory in the computer used precluded the possibility of storing such tables and interpolating.

The functions actually required were $Z_i(x)$ and $Z_i'(x)$ $(1 \leqq i \leqq 4)$, where

$$Z_1(x) = \text{ber } x$$

$$Z_2(x) = -\text{bei } x$$

$$Z_3(x) = -\frac{2}{\pi} \text{kei } x$$

$$Z_4(x) = -\frac{2}{\pi} \text{ker } x;$$

Received Feburary 5, 1962.