R. M. Robinson, whose IBM 701 program produced the factorization (5), and to D. H. Lehmer, whose suggestions have materially assisted in the planning of this work.

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2. JOHN BRILLHART, “Concerning the numbers $2^p + 1$, $p$ prime,” Math. Comp., v. 16, 1962, p. 424–430. (Reference is there made to the earlier table by M. Kraitchik.)

EDITORIAL NOTE: Rudolph Ondrejka has shown that $10^{67} - 1$ is divisible by 2028119. (Recreational Math. Mag., Feb. 1962, p. 47.)

A Note on Octic Permutation Polynomials

By S. R. Cavior

1. Introduction. A polynomial $f(x)$ with coefficients in the finite field $GF(q)$, $q = p^n$, is called a permutation polynomial if the set $\{f(a): a \in GF(q)\}$ is a permutation of $GF(q)$. The object of this paper is to extend some known results about permutation polynomials of even degree over fields with odd characteristic $p$.

We shall frequently use the following theorem which is given by Dickson [1, p. 77].

Theorem. If $f(x)$ is a polynomial of degree $m$ over $GF(q)$, and if $m | q - 1$, then $f(x)$ does not permute $GF(q)$.

To begin our discussion, we note immediately, by the Theorem, that a quadratic polynomial cannot permute $GF(q)$. Dickson, in [1], showed that a quartic cannot permute $GF(q)$ for $q > 7$ (although two do for $q = 7$), and that a sextic cannot permute $GF(q)$ for $q > 11$ (although several do for $q = 11$.) A natural question to ask, then, is whether there is an upper bound for the order of a finite field which an octic can permute.

The present investigation, however, is restricted to the following special octics:

(1) $f(x) = x^8 + ax^t \quad t = 1, 3, 5, 7; a \in GF(q)$.

The case $t = 7$ can be settled at once, for if $f(x) = x^8 + ax^7$, where $a \in GF(q)$, then $f(-a) = f(0) = 0$. That is, $f(x)$ is not a permutation polynomial. With the aid of a computer it was discovered that the only polynomials of the form (1) which permute $GF(p)$ for $p < 500$ are

(2) $x^8 + ax \quad a = \pm 4, \pm 10; p = 29$

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and

\[ x^8 + ax^3 \quad a = \pm 4, \pm 9; p = 11. \]

2. Dickson's Method. The method we use to decide whether a polynomial of the form (1) permutes $GF(q)$ is the one Dickson used in [1]. The basis of it is this fact: If $f(x)$ is a permutation polynomial over $GF(q)$, and is raised to a power less than $q - 1$, the coefficient of $x^{q-1}$ becomes 0 after reducing exponents by the identity $x^q = x$. Therefore, to demonstrate that $f(x)$ is not a permutation polynomial, one must simply show that when it is raised to some (well chosen) power, $x^{q-1}$ does not vanish.

For example, let us take $f(x) = x^8 + ax$ over the field $GF(q), q = 8m + 5$. Raising to the power $(m + 4)$, we have

\[ (x^8 + ax)^{m+4} = x^{8m+32} + a \left(\frac{m+4}{m+3}\right)x^{9m+25} + a^2 \left(\frac{m+4}{m+2}\right)x^{8m+18} \]

\[ + a^3 \left(\frac{m+4}{m+1}\right)x^{8m+11} + a^4 \left(\frac{m+4}{m}\right)x^{8m+4} + \cdots \]

For $q > 29$ none of the exponents can reduce to $8m + 4$ by the identity $x^q = x$. Therefore, if $f(x)$ is to be a permutation polynomial over $GF(q)$, the coefficient of $x^{8m+4}$ must be 0; i.e.,

\[ a^4 \left(\frac{m+4}{m}\right) = 0 \mod p \quad \text{or} \quad p \mid a^4(m+4)(m+3)(m+2)(m+1). \]

However, we shall show that this is impossible if $a \neq 0$. First, $p \nmid m + 1$. For if $p \mid m + 1$, then $p \mid 8m + 8 = p^n + 3$, and $p \mid 3$. But $p = 8t + 5$, so $p \nmid 3$. In a similar way we can show that $p \nmid m + 2, p \nmid m + 3, \text{and} p \nmid m + 4$. So $p \mid a$. This shows, then, that $x^8 + ax$ cannot permute $GF(q)$ if $q = 8m + 5 > 29$.

3. Results. Combining the results in (2) and (3) with other results derived by Dickson's method, we present the following information which indicates upper bounds for the size $q$ of a finite field which the special octics permute.

The polynomial $f(x) = x^8 + ax, a \in GF(q)$, does not permute $GF(q)$ if $q = 8m + 3$ or $8m + 7$. If $q = 8m + 5$ the only field permuted is $GF(29)$.

The polynomial $g(x) = x^8 + ax^3$ does not permute $GF(q)$ if $q = 8m + 5$ or $8m + 7$. If $q = 8m + 3$, and if some $g(x)$ permutes $GF(q)$, then $q$ must equal $11^n$. By the Theorem we see that no octic can permute $GF(11^{2m})$, and it is an open question whether $g(x)$ can permute $GF(11^{2m+1})$.

The polynomial $h(x) = x^8 + ax^5$ does not permute $GF(q)$ if $q = 8m + 3$. If $q = 8m + 5, \text{and if } h(x) \text{ permutes } GF(q), \text{then } q = 13^n$. By the Theorem we see that no octic can permute $GF(13^{2m})$, and it is an open question whether $h(x)$ can permute $GF(13^{2m+1})$. If $q = 8m + 7, \text{and if } h(x) \text{ permutes } GF(q), \text{then } q = 7^n$. Again we see by the Theorem that no octic can permute $GF(7^{2m})$, and again it remains an open question whether $h(x)$ can permute $GF(7^{2m+1})$.  

We now present these results in tabular form.

<table>
<thead>
<tr>
<th>polynomial</th>
<th>$q = p^n$</th>
<th>$GF(q)$ which are permuted</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^8 + ax$</td>
<td>8m + 3</td>
<td>none</td>
</tr>
<tr>
<td></td>
<td>8m + 5</td>
<td>$GF(29)$ and no others</td>
</tr>
<tr>
<td></td>
<td>8m + 7</td>
<td>none</td>
</tr>
<tr>
<td>$x^8 + ax^3$</td>
<td>8m + 3</td>
<td>$GF(11)$ and possibly $GF(11^n)$ for odd $n$</td>
</tr>
<tr>
<td></td>
<td>8m + 5</td>
<td>none</td>
</tr>
<tr>
<td></td>
<td>8m + 7</td>
<td>none</td>
</tr>
<tr>
<td>$x^8 + ax^5$</td>
<td>8m + 3</td>
<td>none</td>
</tr>
<tr>
<td></td>
<td>8m + 5</td>
<td>possibly $GF(13^n)$ for odd $n$</td>
</tr>
<tr>
<td></td>
<td>8m + 7</td>
<td>possibly $GF(7^n)$ for odd $n$</td>
</tr>
</tbody>
</table>

In conclusion we might ask whether, for each integer $k$, there exists a bound $N = N_k$ such that if $f(x)$ is of degree $2k$ over $GF(q)$, $f(x)$ will not permute $GF(q)$ if $q > N_k$.

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**Multistep Integration Formulas**

*By A. C. R. Newbery*

A multistep formula for the approximate solution of an ordinary differential equation $x' = f(x, t)$ has the form $\sum_{i=0}^{k} a_i x_i' = h \sum_{i=0}^{k} b_i x_i'$. The formula is assumed to be stable, and to have optimum precision subject to this restriction; this means that a truncation error of the form $Hh^{k+2}x^{(k+2)}(x) + 0(h^{k+3})$ is associated with the formula [1], where $x(t)$ is the exact solution of the differential equation and $H$ is a constant, which, like the $b_i$, depends on the choice of the constants $a_i$. A closed expression for the $b_i$ has already been given in [2, page 39], but it is considered worthwhile to tabulate the matrices which transform the $a_i$ into the $b_i$, to give an improved derivation of these matrices, and to extend the argument so that predictor coefficients can also be readily calculated.

The first task is, for a given $k$, to compute the elements $c_{ij}$ of a $(k + 2) \times k$ 'corrector matrix' $C_k$, such that $b = C_k a$, where $b = \{b_0, b_1, \cdots b_k, H\}$ and $a = \{a_1, a_2, \cdots a_k\}'$. (Note that $a_0$ is determined by the consistency condition $\sum_{i=0}^{k} a_i = 0$.) Using the notation of Antosiewicz and Gautschi [4, page 327] the relation between the required $b_i$ and the given $a_i$ is equivalent to the requirement that the linear functional $Lx(t) = \sum_{i=0}^{k} [a_i x(i) - b_i x'(i)]$ should annihilate all

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