We now present these results in tabular form.

polynomial	$q = p^n$	$GF(q)$ which are permuted
$x^8 + ax$	$8m + 3$	none
	$8m + 5$	$GF(29)$ and no others
	$8m + 7$	none
$x^8 + ax^3$	$8m + 3$	$GF(11)$ and possibly $GF(11^n)$ for odd n
	$8m + 5$	none
	$8m + 7$	none
$x^8 + ax^5$	$8m + 3$	none
	$8m + 5$	possibly $GF(13^n)$ for odd n
	$8m + 7$	possibly $GF(7^n)$ for odd n

In conclusion we might ask whether, for each integer k, there exists a bound $N = N_k$ such that if $f(x)$ is of degree $2k$ over $GF(q)$, $f(x)$ will not permute $GF(q)$ if $q > N_k$.

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Multistep Integration Formulas

By A. C. R. Newbery

A multistep formula for the approximate solution of an ordinary differential equation $x' = f(x, t)$ has the form $\sum_{i=0}^{k} a_i x_i = h \sum_{i=0}^{k} b_i x_i'$. The formula is assumed to be stable, and to have optimum precision subject to this restriction; this means that a truncation error of the form $Hh^{k+2}x^{(k+2)}(z) + 0(h^{k+3})$ is associated with the formula [1], where $x(t)$ is the exact solution of the differential equation and H is a constant, which, like the b_i, depends on the choice of the constants a_i. A closed expression for the b_i has already been given in [2, page 39], but it is considered worthwhile to tabulate the matrices which transform the a_i into the b_i, to give an improved derivation of these matrices, and to extend the argument so that predictor coefficients can also be readily calculated.

The first task is, for a given k, to compute the elements c_{ij} of a $(k + 2) \times k$ 'corrector matrix' C_k, such that $b = C_k a$, where $b = \{b_0, b_1, \cdots b_k, H\}'$ and $a = \{a_1, a_2, \cdots a_k\}'$. (Note that a_0 is determined by the consistency condition $\sum_0^k a_i = 0$.) Using the notation of Antosiewicz and Gautschi [4, page 327] the relation between the required b_i and the given a_i is equivalent to the requirement that the linear functional $Lx(t) \equiv \sum_{i=0}^{k} [a_i x(i) - b_i x'(i)]$ should annihilate all

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polynomials $x(t)$ of degree $\leqq k + 1$. We define

$$(1)\qquad \begin{aligned} \pi(t) &= \prod_{i=0}^{k} (t - i), & \pi_i(t) &= \pi(t)/(t - i), \\ \bar{\pi}(t) &= \int_0^t \pi(u)\, du, & \bar{\pi}_i(t) &= \int_0^t \pi_i(u)\, du. \end{aligned}$$

Then for $i, j \in \{0, 1, \cdots k\}$ we have

$$\bar{\pi}_j'(i) = \pi_j(i) = 0 \quad (i \neq j), \qquad \bar{\pi}_j'(j) = \pi'(j),$$

so that

$$L\bar{\pi}_j(t) = \sum_{i=0}^{k} a_i \bar{\pi}_j(i) - b_j \pi'(j).$$

Since $\bar{\pi}_j(t)$ is a polynomial of degree $k + 1$, it must be annihilated by L; hence

$$(2)\qquad b_j = \frac{1}{\pi'(j)} \sum_{i=1}^{k} a_i\, \bar{\pi}_j(i).$$

(Since $\bar{\pi}_j(0) = 0$, the lower limit of summation may be taken as 1). In order to determine H, we set $x(t) = \bar{\pi}(t)$, so that $x'(t) = \pi(t)$ and $x^{(k+2)}(t) = (k + 1)!$. When we form $L\bar{\pi}(x)$, we note that all the terms involving b_i vanish; consequently

$$(3)\qquad H = \frac{1}{(k+1)!} L\bar{\pi}(t) = \frac{1}{(k+1)!} \sum_{i=1}^{k} a_i\, \bar{\pi}(i).$$

Combining the results (2), (3) into matrix form, we have for a given k,

$$\begin{aligned} &b = Ca, \qquad a = \{a_1, a_2, \cdots a_k\}', \quad b = \{b_0, b_1, \cdots b_k, H\}, \\ (4)\quad &C = [c_{ij}], \qquad c_{ij} = \bar{\pi}_i(j)/\pi'(i) \quad (0 \leqq i \leqq k, 1 \leqq j \leqq k), \\ &\qquad\qquad\qquad\qquad c_{k+1,j} = \bar{\pi}(j)/(k + 1)! \end{aligned}$$

A 'predictor matrix' can be similarly derived; the vector b is now defined by $b = \{b_0, b_1, \cdots b_{k-1}, H\}$, L is subject to the restriction $b_k = 0$, and in (1) we define $\pi(t) = \prod_{i=0}^{k-1} (t - i)$. For a given k the matrix P is of dimension $(k + 1) \times k$. The result is

$$\begin{aligned} &b = Pa, \qquad p_{ij} = \bar{\pi}_i(j)/\pi'(i) \quad (0 \leqq i \leqq k - 1, 1 \leqq j \leqq k), \\ &\qquad\qquad\qquad\qquad p_{kj} = \bar{\pi}(j)/k! \end{aligned}$$

The matrix elements c_{ij}, p_{ij} have been calculated exactly in rational arithmetic for $k = 2 \cdots 8$, and the results are tabulated below. In order to avoid tabulating fractional elements, the lowest common denominator D has been written above each matrix; thus for $k = 2$, $c_{12} = \frac{32}{24}$. These matrices provide a compact tabulation of all the standard multistep formulas; if one ignores the last row of each corrector matrix, the last column gives the Newton-Cotes coefficients; the difference of the last two columns gives the Adams coefficients; various linear combinations of the columns give coefficients for the various radial and other formulas discussed in [3].

Predictor Matrices

$K = 2, D = 12$

6	0
6	24
−1	4

$K = 3, D = 24$

10	8	18
16	32	0
−2	8	54
1	0	9

$K = 4, D = 720$

270	240	270	0
570	960	810	1920
−150	240	810	−960
30	0	270	1920
−19	−8	−27	224

$K = 5, D = 1440$

502	464	486	448	950
1292	1984	1836	2048	−500
−528	384	1296	768	6000
212	64	756	2048	−3500
−38	−16	−54	448	4250
27	16	27	0	475

$K = 6, D = 60480$

19950	18816	19278	18816	19950	0
59934	86688	82782	86016	78750	199584
−33516	9408	43092	32256	52500	−254016
20244	9408	43092	86016	52500	471744
−7266	−4032	−7938	18816	78750	−254016
1134	672	1134	0	19950	199584
−863	−592	−783	−512	−1375	17712

$K = 7, D = 120960$

38174	36448	36990	36608	37150	35424	73598
130224	180480	174960	178176	174000	186624	−82320
−92922	1056	62694	49152	63750	23328	837606
75008	42496	117504	192512	160000	235008	−1141504
−40422	−25824	−39366	22272	116250	23328	1434426
12624	8448	11664	6144	56400	186624	−707952
−1726	−1184	−1566	−1024	−2750	35424	432866
1375	1024	1215	1024	1375	0	36799

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1} \\ H \end{bmatrix} = P_k \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

$K = 8, D = 3628800$

1103970	1062720	1073250	1067520	1073250	1062720	1103970	0
4195470	5629440	5503950	5560320	5508750	5598720	5258190	14131200
−3653910	−613440	1115370	829440	1046250	699840	1944810	−29306880
3693990	2350080	4800870	6850560	6243750	7050240	4393830	67461120
−2656410	−1849920	−2456730	−407040	2043750	699840	4393830	−75540480
1244970	898560	1115370	829440	2558250	5598720	1944810	67461120
−340530	−250560	−302130	−245760	−371250	1062720	5258190	−29306880
41250	30720	36450	30720	41250	0	1103970	14131200
−33953	−26656	−29889	−27392	−30625	−23328	−57281	1012736

Corrector Matrices

$K = 2, D = 24$

10	8
16	32
−2	8
1	0

$K = 3, D = 720$

270	240	270
570	960	810
−150	240	810
30	0	270
−19	−8	−27

$K = 4, D = 1440$

502	464	486	448
1292	1984	1836	2048
−528	384	1296	768
212	64	756	2048
−38	−16	−54	448
27	16	27	0

$K = 5, D = 60480$

19950	18816	19278	18816	19950
59934	86688	82782	86016	78750
−33516	9408	43092	32256	52500
20244	9408	43092	86016	52500
−7266	−4042	−7938	18816	78750
1134	672	1134	0	19950
−863	−592	−783	−512	−1375

$K = 6, D = 120960$

38174	36448	36990	36608	37150	35424
130224	180480	174960	178176	174000	186624
−92922	1056	62694	49152	63750	23328
75008	42496	117504	192512	160000	235008
−40422	−25824	−39366	22272	116250	23328
12624	8448	11664	6144	56400	186624
−1726	−1184	−1566	−1024	−2750	35424
1375	1024	1215	1024	1375	0

$K = 7,\ D = 3628800$

1103970	1062720	1073250	1067520	1073250	1062720	1103970
4195470	5629440	5503950	5560320	5508750	5598720	5258190
−3653910	−613440	1115370	829440	1046250	699840	1944810
3693990	2350080	4800870	6850560	6243750	7050240	4393830
−2656410	−1849920	−2456730	−407040	2043750	699840	4393830
1244970	898560	1115370	829440	2558250	5598720	1944810
−340530	−250560	−302130	−245760	−371250	1062720	5258190
41250	30720	36450	30720	41250	0	1103970
−33953	−26656	−29889	−27392	−30625	−23328	−57281

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \\ H \end{bmatrix} = C_k \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

$K = 8,\ D = 7257600$

2140034	2072128	2086722	2080256	2085250	2078784	2093378	2025472
8934188	11685376	11486124	11558912	11507500	11570688	11432876	12058624
−9209188	−2719616	556956	124928	377500	93312	681884	−1900544
11190716	7685632	12949308	16769024	15917500	16713216	15203132	21495808
−10066240	−7431680	−9097920	−4648960	−200000	−1866240	768320	−9297920
6292676	4782592	5578308	4726784	8546500	13810176	10305092	21495808
−2582428	−1993856	−2278044	−2025472	−2457500	819072	7308644	−1900544
625748	487936	551124	499712	572500	373248	3124436	12058624
−67906	−53312	−59778	−54784	−61250	−46656	−114562	2025472
57281	46656	50625	48128	50625	46656	57281	0

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On the Non-Existence of Fibonacci Squares

By M. Wunderlich

The Fibonacci sequence, F_n, is defined as follows:

$$F_1 = 1; \qquad F_2 = 1; \qquad F_n = F_{n-2} + F_{n-1} \qquad \text{for} \quad n > 2. \tag{1}$$

A. P. Rollet [1] has posed the following problem. There are only three known Fibonacci numbers which are squares; $F_1 = 1$, $F_2 = 1$, and $F_{12} = 144$. Are there any others? The purpose of this note is to announce that except for the known cases, F_n cannot be a square for $n \leqq 1{,}000{,}000$, and to describe the computational method used to arrive at this result. The referee has kindly pointed out that the method used is somewhat analogous to familiar "exclusion" methods such as those described in [2].

Let p be an arbitrary fixed prime number. With respect to this prime, denote by

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