Jacobi Polynomial Expansions of a Generalized Hypergeometric Function over a Semi-Infinite Ray

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1. Introduction. Suppose $f(x)$ is continuous and has a piecewise continuous derivative for $0 \leqq x/\lambda \leqq 1$. Then $f(x)$ may be expanded into a uniformly convergent series of shifted Jacobi polynomials in the form

$$f(x) = \sum_{n=0}^{\infty} a_n(\lambda) R_n^{(\alpha,\beta)}(x/\lambda), \tag{1.1}$$

$$\epsilon \leqq x/\lambda \leqq 1 - \epsilon, \quad \epsilon > 0; \quad \alpha > -1, \quad \beta > -1,$$

where $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x - 1)$ and the latter is the usual notation for the Jacobi polynomial [1, Ch. 10]. Various techniques are available for the determination of the coefficients $a_n(\lambda)$. In this connection, see, for example, the references [2, 3, 4, 5, 6, 7].

Suppose that $f(x)$ satisfies the above conditions for $1 \leqq x/\lambda \leqq \infty$ where $|\arg \lambda| < \varphi$. Then we may write

$$f(x) = \sum_{n=0}^{\infty} b_n(\lambda) R_n^{(\alpha,\beta)}(\lambda/x), \tag{1.2}$$

$$\epsilon \leqq \lambda/x \leqq 1 - \epsilon, \quad \epsilon > 0; \quad \alpha > -1, \quad \beta > -1.$$

If $f(x)$ has an asymptotic expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad x \to \infty, \qquad |\arg x| < \varphi, \tag{1.3}$$

then (1.2) may be interpreted as a summability process which converts the generally divergent expansion (1.3) into a convergent expansion. If $f(x)$ in (1.3) is of hypergeometric type,* then the coefficients $b_n(\lambda)$ may be found formally at least using the procedures [5, 6]. These yield for $b_n(\lambda)$ an asymptotic series in λ which is also of hypergeometric type. The asymptotic representation for $b_n(\lambda)$ in general is not suitable for computation. We are confronted with two problems: one is the interpretation of the asymptotic series for $b_n(\lambda)$, and the other is the computation of $b_n(\lambda)$.

In this paper, we show how both problems can be solved for a confluent hypergeometric function. Actually we derive a representation for $b_n(\lambda)$ when $f(x)$ is the G-function, which includes the confluent hypergeometric function as a special case. Our computational scheme for $b_n(\lambda)$ is exhibited only when $f(x)$ is a confluent hypergeometric function, although the ideas involved can be extended to cover other special cases of the G-function as well.

In Section II, we prove an expansion theorem of the form (1.2) when $f(x)$ is the

Received March 29, 1963.

* For the definition and properties of generalized hypergeometric series including the G-function as well as other notations used in this paper, see [1, Chs. 4, 5, 6].

G-function and show how both convergent and asymptotic representations for $b_n(\lambda)$ may be derived. These results are specialized in Section III for the case when $f(x)$ is a confluent hypergeometric function, and in Section IV it is shown how $b_n(\lambda)$ may be computed by a recursion scheme. Finally, in Section V, we tabulate coefficients for the cases where $R_n^{(\alpha,\beta)}(x)$ is the shifted Chebyshev polynomial and $f(x)$ is the error function, the exponential, sine and cosine integrals, and the Bessel functions $K_0(x)$ and $K_1(x)$.

2. Expansion of the G-Function. The G-function is given by the Mellin-Barnes integral

$$(2.1)\quad G_{p,q}^{m,k}\left(\lambda x \,\middle|\, {a_p \atop b_q}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{k}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=k+1}^{p}\Gamma(a_j - s)}(\lambda x)^s\,ds,$$

where an empty product is interpreted as 1, $0 \leqq m \leqq q$, $0 \leqq n \leqq p$ and the parameters are such that no pole of $\Gamma(b_j - s)$, $j = 1, 2, \cdots, m$ coincides with any pole of $\Gamma(1 - a_h + s)$, $h = 1, 2, \cdots, k$. We assume x is real and the path L runs parallel to the imaginary axis and is indented so that the poles of $\Gamma(b_j - s)$, $j = 1, 2, \cdots, m$, are to the right, and all the poles of $\Gamma(1 - a_h + s)$, $h = 1, 2, \cdots, k$, to the left of L. The integral converges if $p + q < 2(m + k)$ and $|\arg\lambda| < (m + k - p/2 - q/2)\pi$. For a treatment of the G-function, see [1, Ch. 5].

Now from [1, 10.20(3)] we have the expansion

$$(2.2)\quad \begin{aligned} x^s &= \Gamma(\beta - s + 1)\Gamma(1-s) \\ &\times \sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)(n+\beta+1)_\alpha}{\Gamma(n+\alpha+\beta+2-s)\Gamma(1-s-n)} R_n^{(\alpha,\beta)}(1/x), \quad 1 < x < \infty, \end{aligned}$$

uniformly for $\operatorname{Re}(s) \leqq \mu - \delta$, $\delta > 0$, $\mu = \min(\beta + 1, \beta/2 + \frac{3}{4})$, $\alpha > -1$, $\beta > -1$. Put (2.2) in (2.1) and integrate along the path from $\mu - \delta - i\infty$ to $\mu - \delta + i\infty$. We then get

THEOREM I. *Let*

1. *α, β and x be real, $\alpha > -1$, $\beta > -1$, $1 < x < \infty$.*

Let a real positive δ exist such that

2. (a) $\operatorname{Re}(a_j - 1) < \mu - \delta$, $j = 1, 2, \cdots k$; (b) $\operatorname{Re}(b_j) > \mu - \delta$, $j = 1, 2, \cdots m$, $\mu - \delta < 1$, $\mu = \min(\beta + 1, \beta/2 + \frac{3}{4})$.

3. $p + q < 2(m + k)$, $|\arg\lambda| < (m + k - p/2 - q/2)\pi$, $\lambda \neq 0$, $0 \leqq m \leqq q$, $0 \leqq k \leqq p$.

Then

$$(2.3)\quad \begin{aligned} G_{p,q}^{m,k}\left(\lambda x_b^a \,\middle|\, {p \atop q}\right) &= \sum_{n=0}^{\infty}(2n+\alpha+\beta+1)(n+\beta+1)_\alpha \\ &\times G_{p+2,q+2}^{m+2,k}\left(\lambda \,\middle|\, {a_p, 1-n, n+\alpha+\beta+2 \atop 1, \beta+1, b_q}\right) R_n^{(\alpha,\beta)}(1/x). \end{aligned}$$

Remark. Assumptions 2 above insure the separation of poles and specify the regions in which they lie according to the remarks surrounding (2.1). Notice, however, that poles of $\Gamma(b_j - s)$ may lie to the left of the contour. They may be excluded by indentations since they lie in a region where the series for x^s converges uniformly, provided they do not coincide with any of the poles of $\Gamma(1 - a_h + s)$. Hence, we may replace 2(b) by the weaker but more complicated condition

2(b)* $1 + \delta_{j-2} - a_h \neq 0, -1, -2, \cdots,$

$$j = 1, 2, \cdots m + 2, h = 1, 2, \cdots k, \delta_{-2} = 1, \delta_{-1} = \beta + 1, \delta_{j-2} = b_j, j > 1.$$

Notice from the definition of the G-function that

$$G^{m+2,k}_{p+2,q+2}\left(\lambda \,\middle|\, {}^{a_p,1-n,n+\alpha+\beta+2}_{1,\beta+1,b_q}\right) = (-)^n G^{m+1,k+1}_{p+2,q+2}\left(\lambda \,\middle|\, {}^{1-n,a_p,n+\alpha+\beta+2}_{\beta+1,b_q,1}\right). \tag{2.4}$$

If $|\arg \lambda| < \frac{1}{2}(p - q + 1)\pi$, an asymptotic representation for the coefficients of $R_n^{(\alpha,\beta)}(1/x)$ in (2.3) follows by application of a result in [1, 5.3(6)]. An ascending series representation follows when [1, 5.3(5)] is applied to the right-hand side of (2.4).

3. Expansion of a Confluent Hypergeometric Function. We consider the function [1, Ch. 6],

$$(\lambda x)^a \psi(a, c \mid \lambda x) = \{\Gamma(a)\Gamma(\sigma)\}^{-1} G^{2,1}_{1,2}\left(\lambda x \,\middle|\, {}^{1}_{a,\sigma}\right), \quad \sigma = a + 1 - c. \tag{3.1}$$

Also, denote by $T_n^*(x)$ the shifted Chebyshev polynomial

$$T_n^*(x) = T_n(2x - 1) = \frac{n!}{(\frac{1}{2})_n} R_n^{(-1/2,-1/2)}(x). \tag{3.2}$$

From Theorem I, we get

THEOREM II. *Let*

1. $1 \leqq x \leqq \infty$;
2. $\sigma \neq 0, -1, -2, \cdots$; $a \neq 0, -1, -2, \cdots$;
3. $|\arg \lambda| < 3\pi/2, \lambda \neq 0$.

Then

$$(\lambda x)^a \psi(a, c \mid \lambda x) = \sum_{n=0}^{\infty} C_n(\lambda) T_n^*(1/x), \tag{3.3}$$

where

$$C_n(\lambda) = \frac{\epsilon_n}{\pi^{1/2}\Gamma(a)\Gamma(\sigma)} G^{4,1}_{3,4}\left(\lambda \,\middle|\, {}^{1,1-n,n+1}_{1,1/2,a,\sigma}\right), \quad \epsilon_0 = 1, \epsilon_n = 2, n > 0, \tag{3.4}$$

or

$$C_n(\lambda) = \frac{\epsilon_n(-)^n}{\pi^{1/2}\Gamma(a)\Gamma(\sigma)} G^{3,1}_{2,3}\left(\lambda \,\middle|\, {}^{1-n,n+1}_{1/2,a,\sigma}\right). \tag{3.5}$$

Also, if none of the quantities $\frac{1}{2}$, a and σ differ by an integer

$$\begin{aligned} C_n(\lambda) = \frac{\epsilon_n(-)^n}{\pi^{1/2}} \Big\{ & (a)_{-1/2}(\sigma)_{-1/2}\lambda^{1/2} {}_2F_2\left({}^{n+1/2,-n+1/2}_{3/2-a,3/2-\sigma} \,\middle|\, \lambda\right) \\ & + \frac{\Gamma(\frac{1}{2} - a)(a)_n(\sigma)_{-a}}{\Gamma(n + 1 - a)} \lambda^a {}_2F_2\left({}^{n+a,-n+a}_{a+1/2,a-\sigma+1} \,\middle|\, \lambda\right) \\ & + \frac{\Gamma(\frac{1}{2} - \sigma)(\sigma)_n(a)_{-\sigma}}{\Gamma(n - \sigma + 1)} \lambda^\sigma {}_2F_2\left({}^{n+\sigma,-n+\sigma}_{\sigma+1/2,\sigma-a+1} \,\middle|\, \lambda\right) \Big\}, \end{aligned} \tag{3.6}$$

and

$$(3.7)\quad C_n(\lambda) \sim \frac{\epsilon_n(-)^n(a)_n(\sigma)_n}{n!(4\lambda)^n}\, {}_3F_1\left(\begin{matrix} n+1/2, n+a, n+\sigma \\ 2n+1 \end{matrix}\middle| -\frac{1}{\lambda}\right), \quad |\lambda| \to \infty, \quad |\arg \lambda| < \pi.$$

Remark. Condition 1 of Theorem I is conservative. By an appeal to the convergence properties of expansions in Chebyshev polynomials [7], the range of x may be extended to give condition 1 above.

Since (3.3) converges,

$$(3.8)\qquad \lim_{n\to\infty} C_n(\lambda) = 0.$$

For later use, we record the fact that

$$(3.9)\qquad \lim_{x\to\infty} (\lambda x)^a \psi(a, c \mid \lambda x) = 1, \qquad |\arg \lambda| < 3\pi/2.$$

4. Calculation of the Coefficients $C_n(\lambda)$.

Let

$$(4.1)\qquad \varphi_{1,n}(\lambda) = \frac{(-)^n}{\epsilon_n} C_n(\lambda).$$

Following the method developed in [8], we can show from the representation (3.7) that $\varphi_{1,n}(\lambda)$ satisfies the recursion relation

$$(4.2)\quad \varphi_n(\lambda) + (A_n + B_n\lambda)\varphi_{n+1}(\lambda) + (C_n + D_n\lambda)\varphi_{n+2}(\lambda) + E_n\varphi_{n+3}(\lambda) = 0,$$

where

$$(4.3)\qquad \begin{aligned} A_n &= (2n+2)\left[1 - \frac{(n+\frac{3}{2})(n+a+1)(n+\sigma+1)}{(n+2)(n+a)(n+\sigma)}\right], \\ B_n &= D_n = -4(n+1)/(n+a)(n+\sigma), \\ C_n &= -1 + [2(n+1)(2n+3)/(n+a)(n+\sigma)], \\ E_n &= -(n+1)(n-a+3)(n-\sigma+3)/(n+2)(n+a)(n+\sigma). \end{aligned}$$

We prove that the coefficients may be readily evaluated using (4.2) in the backward direction. This backward recursion technique has been treated by many authors [9], [10], [11], [12], [13]. The idea is as follows.

For fixed λ, arbitrary η and ν sufficiently large set

$$(4.4)\qquad \varphi_\nu^{(\nu)}(\lambda) = \varphi_{\nu-1}^{(\nu)}(\lambda) = 0,$$

$$(4.5)\qquad \varphi_{\nu-2}^{(\nu)}(\lambda) = \eta.$$

The sequence $\varphi_{\nu-3}^{(\nu)}(\lambda), \cdots, \varphi_n^{(\nu)}(\lambda), \cdots \varphi_1^{(\nu)}(\lambda), \varphi_0^{(\nu)}(\lambda)$ is generated from (4.2). Using (3.9) and

$$(4.6)\qquad T_n^*(0) = (-)^n$$

in (3.3) we would expect that if

$$(4.7)\qquad \omega_\nu = \sum_{n=0}^{\nu-2} \epsilon_n \varphi_n^{(\nu)}(\lambda),$$

then

$$C_n(\lambda) \sim (-)^n \epsilon_n \varphi_n^{(\nu)}(\lambda)/\omega_\nu , \tag{4.8}$$

with increasing accuracy as $\nu \to \infty$. In fact if we define

$$\varphi_{1,n}^{(\nu)}(\lambda) = \varphi_{1,0}(\lambda)\varphi_n^{(\nu)}(\lambda)/\varphi_0^{(\nu)}(\lambda), \tag{4.9}$$

we have:

THEOREM III. *Let* $|\arg \lambda| < \pi$, $\lambda \neq 0$, *and neither* a *nor* σ *be a negative integer or zero. Then*

$$\lim_{\nu\to\infty} \varphi_{1,n}^{(\nu)}(\lambda) = \varphi_{1,n}(\lambda). \tag{4.10}$$

Proof. Denote by $\varphi_{1,n}(\lambda)$, $\varphi_{2,n}(\lambda)$ and $\varphi_{3,n}(\lambda)$ the three linearly independent solutions of (4.2); $\varphi_{1,n}(\lambda)$ is the solution we wish to calculate. We may write*

$$\varphi_n^{(\nu)} = \xi_1^{(\nu)}\varphi_{1,n} + \xi_2^{(\nu)}\varphi_{2,n} + \xi_3^{(\nu)}\varphi_{3,n} , \qquad n < \nu - 2, \tag{4.11}$$

and the conditions (4.4) and (4.5) give

$$0 = \xi_1^{(\nu)}\varphi_{1,\nu} + \xi_2^{(\nu)}\varphi_{2,\nu} + \xi_3^{(\nu)}\varphi_{3,\nu} , \tag{4.12}$$

$$0 = \xi_1^{(\nu)}\varphi_{1,\nu-1} + \xi_2^{(\nu)}\varphi_{2,\nu-1} + \xi_3^{(\nu)}\varphi_{3,\nu-1} , \tag{4.13}$$

$$\eta = \xi_1^{(\nu)}\varphi_{1,\nu-2} + \xi_2^{(\nu)}\varphi_{2,\nu-2} + \xi_3^{(\nu)}\varphi_{3,\nu-2} , \tag{4.14}$$

where $\xi_1^{(\nu)}$, $\xi_2^{(\nu)}$ and $\xi_3^{(\nu)}$ are independent of n.

$$\xi_2^{(\nu)}/\xi_1^{(\nu)} = \gamma_\nu , \qquad \xi_3^{(\nu)}/\xi_1^{(\nu)} = \delta_\nu , \tag{4.15}$$

$$\gamma_\nu = [-\varphi_{1,\nu}\varphi_{3,\nu-1} + \varphi_{1,\nu-1}\varphi_{3,\nu}]/\tau_\nu , \tag{4.16}$$

$$\delta_\nu = [-\varphi_{2,\nu}\varphi_{1,\nu-1} + \varphi_{1,\nu}\varphi_{2,\nu-1}]/\tau_\nu , \tag{4.17}$$

$$\tau_\nu = [\varphi_{2,\nu}\varphi_{3,\nu-1} - \varphi_{3,\nu}\varphi_{2,\nu-1}]. \tag{4.18}$$

Thus

$$\varphi_{1,n}^{(\nu)} = \frac{\varphi_{1,n}\{1 + (\gamma_\nu\varphi_{2,n}/\varphi_{1,n}) + (\delta_\nu\varphi_{3,n}/\varphi_{1,n})\}}{\{1 + (\gamma_\nu\varphi_{2,n}/\varphi_{1,0}) + (\delta_\nu\varphi_{3,0}/\varphi_{1,0})\}}. \tag{4.19}$$

We will show that

$$\lim_{\nu\to\infty} \gamma_\nu = \lim_{\nu\to\infty} \delta_\nu = 0. \tag{4.20}$$

Equation (3.8) gives

$$\lim_{\nu\to\infty} \varphi_{1,\nu} = 0. \tag{4.21}$$

It may be directly verified that

$$\varphi_{2,n} = {}_2F_2\left({n+1/2,-n+1/2 \atop 3/2-a,3/2-\sigma} \,\middle|\, \lambda\right), \tag{4.22}$$

is also a solution of (4.2). From [14] we have

$$\varphi_{2,n} = C_1 n^{2/3[a+\sigma-2]} \exp\left[\tfrac{3}{2}n^{2/3}\lambda^{1/3}\right]\left[1 + O\left(\frac{1}{n}\right)\right], \qquad |\arg \lambda| < \pi, \tag{4.23}$$

* Henceforth we write, $\xi_1^{(\nu)}(\lambda) = \xi_1^{(\nu)}$, $\varphi_{1,n}(\lambda) = \varphi_{1,n}$, etc.

TABLE I

Coefficients for the Series

$$-Ei(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = \frac{e^{-x}}{x}\sum_{n=0}^{\infty} A_n T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty,$$

$$Erfc(x) = \int_x^\infty e^{-t^2}\,dt = \frac{e^{-x^2}}{2x}\sum_{n=0}^{\infty} B_n T_{2n}\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty.$$

n	A_n	B_n	n	A_n	B_n
0	0.90535 40999 62349 15873 (00)	0.94960 80415 75614 24493 (00)	19	−0.47291 68 (−13)	−0.83782 0 (−14)
1	−0.86481 17855 25987 1490 (−01)	−0.47027 51541 55887 6766 (−01)	20	0.14637 62 (−13)	0.25450 2 (−14)
2	0.72241 01543 74659 475 (−02)	0.30329 41457 65811 336 (−02)	21	−0.46173 9 (−14)	−0.78866 (−15)
3	−0.80975 59457 55738 62 (−03)	−0.29024 22664 89234 09 (−03)	22	0.14827 1 (−14)	0.24900 (−15)
4	0.10999 13443 26613 89 (−03)	0.35220 95068 75272 1 (−04)	23	−0.48417 (−15)	−0.8001 (−16)
5	−0.17173 32998 93776 7 (−04)	−0.50403 49016 87590 (−05)	24	0.16062 (−15)	0.2614 (−16)
6	0.29856 27514 47928 (−05)	0.81642 08349 9637 (−06)	25	−0.5409 (−16)	−0.867 (−17)
7	−0.56596 49145 7719 (−06)	−0.14583 31253 5019 (−06)	26	0.1847 (−16)	0.292 (−17)
8	0.11526 80839 7141 (−06)	0.28219 97993 581 (−07)	27	−0.640 (−17)	−0.100 (−17)
9	−0.24950 30440 269 (−07)	−0.58403 61925 03 (−08)	28	0.224 (−17)	0.35 (−18)
10	0.56923 24201 83 (−08)	0.12803 21362 01 (−08)	29	−0.80 (−18)	−0.12 (−18)
11	−0.13599 57664 81 (−08)	−0.29508 80097 7 (−09)	30	0.29 (−18)	0.4 (−19)
12	0.33846 62888 8 (−09)	0.71082 35787 (−10)	31	−0.10 (−18)	−0.2 (−19)
13	−0.87378 53904 (−10)	−0.17809 90369 (−10)	32	0.4 (−19)	0.1 (−19)
14	0.23315 88663 (−10)	0.46230 4869 (−11)	33	−0.1 (−19)	
15	−0.64114 8105 (−11)	−0.12391 4208 (−11)			
16	0.18122 4698 (−11)	0.34199 361 (−12)			
17	−0.52538 318 (−12)	−0.96955 55 (−13)			
18	0.15592 183 (−12)	0.28176 05 (−13)			

TABLE II

Coefficients for the Series

$$K_0(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{n=0}^{\infty} A_n T_n^*\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty,$$

$$K_1(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{n=0}^{\infty} B_n T_n^*\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty.$$

n	A_n	B_n	n	A_n	B_n
0	0.97354 00764 30036 78069 (+00)	1.08537 28165 51726 12040 (+00)	16	0.64093 20 (−13)	−0.71623 74 (−13)
1	−0.25091 95450 33808 0930 (−01)	0.82919 14491 55864 9624 (−01)	17	−0.17772 98 (−13)	0.19764 83 (−13)
2	0.12525 86114 67721 930 (−02)	−0.22802 07949 89514 525 (−02)	18	0.50586 5 (−14)	−0.56010 2 (−14)
3	−0.10252 45722 44517 42 (−03)	0.15575 94482 17418 05 (−03)	19	−0.14749 6 (−14)	0.16266 5 (−14)
4	0.11130 34099 23675 6 (−04)	−0.15448 62470 24490 3 (−04)	20	0.43980 (−15)	−0.48328 (−15)
5	−0.14615 29450 74297 (−05)	0.19200 97185 68380 (−05)	21	−0.13390 (−15)	0.14666 (−15)
6	0.22075 97885 5320 (−06)	−0.27941 60297 7437 (−06)	22	0.4157 (−16)	−0.4540 (−16)
7	−0.37185 32935 143 (−07)	0.45807 22385 967 (−07)	23	−0.1315 (−16)	0.1431 (−16)
8	0.68410 89366 29 (−08)	−0.82547 24141 10 (−08)	24	0.423 (−17)	−0.459 (−17)
9	−0.13544 36576 47 (−08)	0.16077 77088 92 (−08)	25	−0.138 (−17)	0.150 (−17)
10	0.28543 50058 7 (−09)	−0.33434 19366 8 (−09)	26	0.46 (−18)	−0.50 (−18)
11	−0.63491 57811 (−10)	0.73551 51365 (−10)	27	−0.15 (−18)	0.17 (−18)
12	0.14808 33144 (−10)	−0.16994 68453 (−10)	28	0.5 (−19)	−0.6 (−19)
13	−0.36021 2795 (−11)	0.41008 3914 (−11)	29	−0.2 (−19)	0.2 (−19)
14	0.90986 015 (−12)	−0.10286 1200 (−11)	30	0.1 (−19)	−0.1 (−19)
15	−0.23777 733 (−12)	0.26716 524 (−12)			

TABLE III

Coefficients for the Series

$$Si(x) = \int_x^\infty \frac{\sin t}{t}\,dt = \sum_{n=0}^{\infty} \left\{ \frac{A_n \cos x}{x} + \frac{B_n \sin x}{x} \right\} T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty,$$

$$Ci(x) = \int_x^\infty \frac{\cos t}{t}\,dt = \sum_{n=0}^{\infty} \left\{ \frac{B_n \cos x}{x} - \frac{A_n \sin x}{x} \right\} T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty.$$

n	A_n	B_n	n	A_n	B_n
0	0.96578 82803 51851 83021 (00)	0.10728 86713 38433 09526 (00)	21	0.17469 921 (−12)	−0.81934 01 (−13)
1	−0.43060 83777 85967 3425 (−01)	0.99693 56055 36349 5732 (−01)	22	−0.38470 01 (−13)	0.60587 82 (−13)
2	−0.73143 71174 81046 083 (−02)	−0.81628 39500 94241 970 (−02)	23	0.20193 (−15)	−0.27093 29 (−13)
3	0.14705 23578 98680 654 (−02)	−0.29696 08630 56773 29 (−03)	24	0.53597 7 (−14)	0.88808 3 (−14)
4	−0.98657 68573 27002 1 (−04)	0.22891 94548 45829 18 (−03)	25	−0.35606 0 (−14)	−0.18751 4 (−14)
5	−0.22743 20220 46550 8 (−04)	−0.41721 77635 53092 6 (−04)	26	0.15788 2 (−14)	−0.8110 (−16)
6	0.98240 25732 25254 (−05)	0.21254 28930 87307 (−05)	27	−0.52547 (−15)	0.34480 (−15)
7	−0.18973 43014 87133 (−05)	0.13157 50436 91368 (−05)	28	0.11378 (−15)	−0.22564 (−15)
8	0.10063 43594 1558 (−06)	−0.55848 57495 6974 (−06)	29	0.512 (−17)	0.10258 (−15)
9	0.80819 36482 224 (−07)	0.12353 72625 6029 (−06)	30	−0.2244 (−16)	−0.3571 (−16)
10	−0.38976 28287 529 (−07)	−0.10318 72179 187 (−07)	31	0.1532 (−16)	0.845 (−17)
11	0.10335 65032 550 (−07)	−0.50159 03675 67 (−08)	32	−0.732 (−17)	−0.3 (−19)
12	−0.14104 34487 59 (−08)	0.30915 99889 01 (−08)	33	0.273 (−17)	−0.146 (−17)
13	−0.25232 07840 0 (−09)	−0.10080 57370 10 (−08)	34	−0.73 (−18)	0.110 (−17)
14	0.25699 83132 6 (−09)	0.20289 59643 1 (−09)	35	0.6 (−19)	−0.56 (−18)
15	−0.10597 88925 4 (−09)	−0.27237 6669 (−11)	36	0.9 (−19)	0.23 (−18)
16	0.28970 03157 (−10)	−0.19967 52281 (−10)	37	−0.8 (−19)	−0.7 (−19)
17	−0.41023 1426 (−11)	0.11219 38506 (−10)	38	0.5 (−19)	0.1 (−19)
18	−0.10437 6937 (−11)	−0.40081 1186 (−11)	39	−0.2 (−19)	0.0 (−20)
19	0.10994 1845 (−11)	0.96702 841 (−12)	40	0.1 (−19)	−0.1 (−19)
20	−0.52214 239 (−12)	−0.71861 28 (−13)			

where C_1 is independent of n. The third linearly independent solution of (4.2) is the $L_{2,2}(-\lambda)$ term appearing in [15, 1.3.3(15)] which arises in the asymptotic expansion of (4.22) for large λ. A limit process, explained in [15, 1.3.4] is used to obtain $\varphi_{3,n}$, but our discussion here is necessarily brief. We need only the estimate

$$\varphi_{3,n} = \frac{C_2\Gamma(n + a - 1)\Gamma(n + \sigma - 1)}{(4\lambda)^n n!}\left[1 + O\left(\frac{1}{n^2}\right)\right], \tag{4.24}$$

where C_2 is independent of n. Thus

$$\lim_{\nu\to\infty} | \varphi_{2,\nu} | = \lim_{\nu\to\infty} | \varphi_{3,\nu} | = \infty . \tag{4.25}$$

Also, from (4.23) and (4.24), we have

$$\tau_\nu = -\varphi_{2,\nu}\varphi_{3,\nu}\left[1 + O\left(\frac{1}{\nu}\right)\right]. \tag{4.26}$$

Hence (4.20) is easily shown and the statement (4.10) follows from (4.19).

5. Tables. Tables I–III contain coefficients to 20 D for the expansions of several important cases of the confluent hypergeometric function [1, 6.9]. Coefficients corresponding to different ranges of the independent variable as well as those for other functions, e.g., $J_\nu(x)$ and $Y_\nu(x)$, are under construction and the present tables are selected examples only. The expansions are readily evaluated using a nesting procedure described in [4], [7]. For similar expansions, see [7], and for many Chebyshev expansions of functions over a finite interval, see [2]–[6] and the references given there. The number in parenthesis after each entry in the tables is the power of 10 by which the entry is to be multiplied.

6. Acknowledgment. This paper covers research initiated by the Applied Mathematics Laboratory, David W. Taylor Model Basin, Washington, D. C., under Contract No. Nonr-2638 (00)(X).

The authors acknowledge with thanks the valuable assistance of our colleagues, particularly the suggestions of Jerry Fields. The coefficients tabulated in Section V were computed on the IBM 1620, and the authors are indebted to the experience and skill of Dean Lawrence, who supervised the calculations.

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