The Second-Order Term in the Asymptotic Expansion of $B(x)$

By Daniel Shanks

1. Introduction. It is a well-known theorem of Landau [1], [2], [3], [4] that if $B(x)$ is the number of integers $\leqq x$ that are expressible in the form $u^2 + v^2$, then

$$B(x) \sim \frac{bx}{\sqrt{\log x}} \tag{1}$$

where

$$b = [2 \prod_q (1 - q^{-2})]^{-1/2}, \tag{2}$$

the product being taken over all primes q of the form $4m + 3$. Empirically, the ratio $B(x)\sqrt{\log x}/bx$ approaches unity slowly from above in very much the same way in which $\pi(x) \log x/x$ approaches unity from above.

Ramanujan [5] independently asserted that

$$B(x) = K \int_1^x \frac{du}{\sqrt{\log u}} + O\left(\frac{x}{\log x}\right)^{1/2} \tag{3}$$

where K (his notation) is also given by the right side of (2). Since

$$K \int_1^x \frac{du}{\sqrt{\log u}} = \frac{Kx}{\sqrt{\log x}}\left[1 + \frac{1}{2 \log x} + O\left(\frac{1}{\log^2 x}\right)\right], \tag{4}$$

the ratio $\int_1^x \frac{du}{\sqrt{\log u}} \cdot \sqrt{\log x}/x$ also approaches unity slowly from above, and Ramanujan's assertion at first seems plausible. In the analogous prime number theorem it is well known that $\int_2^x du/\log u$ approximates $\pi(x)$ much better than $x/\log x$ does.

G. H. Hardy [3, p. 9, p. 63] stated, however, that Ramanujan's "integral has no advantage, as an approximation, over the simpler function $Kx/\sqrt{\log x}$." Now empirically, as we shall see, the integral is definitely a closer approximation to $B(x)$. One therefore first assumes that Hardy did not mean to be taken literally here, and that he merely meant that the second-order term in (4) is not the correct one; specifically, that the coefficient $\frac{1}{2}$ is inaccurate. However, upon examination of the original paper [6] of Hardy's student, Miss G. K. Stanley, it was found that she states, in effect, that the correct second-order coefficient is *negative*. If this were true, then Hardy's remark would be entirely unobjectionable, since Ramanujan's integral (4) would, in fact, be *less* accurate than the leading term. Apparently Hardy believed this to be the case, for later he writes [3, p. 19] "The integral is better replaced by the simpler function $\cdots$."

But that is in such conflict with the actual behavior of $B(x)$ that it became apparent that there must be an error in [6]. In fact, there are several errors, and these nullify the proof there that Ramanujan's second term is wrong. Nonetheless, it is wrong, as we shall verify.

Received June 25, 1963.

In the present paper [8] we will correct the several errors in [6], and show how to accurately compute the first two coefficients in

$$(5) \qquad B(x) = \frac{bx}{\sqrt{\log x}}\left[1 + \frac{c}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

We then give a comparison of $B(x)$ with the right sides of (1), (3), and (5). Finally, we prove some related theorems, and, associated with these, we note a simple, elementary argument that Ramanujan could have used (since it does not involve Cauchy's Theorem) to convince himself that his equation (3) was highly improbable.

2. Analysis of the Errors in [6]. Stanley uses the same analysis as in Landau's original paper [1]. Let $b_n = 1$ if $n = u^2 + v^2$ and $b_n = 0$ otherwise. Then $B(x) = \sum_{n \leqq x} b_n$. Let $f(s) = \sum_{n=1}^{\infty} b_n n^{-s}$. Landau proved that $f(s)$ has a branch point at $s = 1$, and a convergent series:

$$(6) \qquad \frac{f(s)}{s^2} = \frac{ai}{\sqrt{1-s}} + a_1 i\sqrt{1-s} + a_2 i(1-s)^{3/2} + \cdots .$$

He further proved, for all m, that

$$(7) \qquad \sum_{n \leqq x} b_n \log \frac{x}{n} = \frac{1}{\pi i}\int_\theta^1 \frac{x^s}{s^2} f(s)\, ds + o\left(\frac{x}{\log^m x}\right),$$

where $0 < \theta < 1$.

From these equations Stanley deduces [6, p. 235] the result:

$$(8) \qquad B(x) = \sum_{n=1}^{x} b_n = \frac{x}{\pi}\left[\frac{a\,\Gamma(\frac{1}{2})}{(\log x)^{1/2}} + \frac{(a_1 - a)\Gamma(\frac{3}{4})}{(\log x)^{3/2}} + O\left(\frac{x}{(\log x)^{7/4}}\right)\right].$$

There is a rather obvious typographical error here but we may correct it without further discussion since no erroneous conclusions were based upon it. The $\Gamma(\frac{3}{4})$ should read $\Gamma(\frac{3}{2})$. In the analysis [6, p. 234] leading to equation (8) there are two other typographical errors. Again, one of these may be changed without discussion, namely, change $\alpha_1 = \frac{1}{2}a/\sqrt{\pi}$ to $\alpha_1 = \frac{1}{2}a_1/\sqrt{\pi}$.

But the other error is important and must be discussed. It reads:

$$(9) \qquad \text{“}\int_\theta^1 x^s(1-s)^{m-1/2}\, ds = \Gamma(m + \tfrac{1}{2})\frac{x}{(\log x)^{n+1/2}} + O\left(\frac{x}{(\log x)^{n+1}}\right)$$

where $\delta > 0$”

It is clear that there is some misprint here, and in a subsequent corrigendum [7] Stanley modified this as follows: change $n + \frac{1}{2}$ to $m + \frac{1}{2}$, $n + 1$ to $m + \frac{3}{2}$, and delete “where $\delta > 0$.” But though this now reads consistently, it does not suffice mathematically. It implies, for $m = 0$, that the error in integrating the leading term in (6) may affect our second-order term by an unknown amount. And since this is the term in question, an error of that order is not acceptable. However, it is easy to prove that

$$\int_\theta^1 x^s(1-s)^{m-1/2}\, ds = \Gamma(m + \tfrac{1}{2})\frac{x}{(\log x)^{m+1/2}} + O\left(\frac{x^{1-\delta}}{(\log x)^{m+1}}\right)$$

for a $\delta > 0$. This was perhaps the original form of (9), prior to printing, and it suffices mathematically for all terms of any order $\frac{x}{(\log x)^r}$. Thus (8), when corrected, namely,

$$(8') \qquad B(x) = \frac{x}{\pi}\left[\frac{a\Gamma(\frac{1}{2})}{(\log x)^{1/2}} + \frac{(a_1 - a)\Gamma(\frac{3}{2})}{(\log x)^{3/2}}\right] + O\left(\frac{x}{(\log x)^{5/2}}\right),$$

is true, and should lead to the correct expansion (5).

But here Stanley makes two nontypographical errors and obtains

$$(10) \qquad \text{“}\frac{a_1}{a} = -2 + \frac{\log 2}{2} - \frac{\sum N^{-2}\log N}{1 + \sum N^{-2}} + \frac{1}{2}\gamma - \frac{2}{\pi}\sum_1^\infty \frac{(-1)^n \log(2n+1)}{2n+1}$$

where N is a prime of the form $4m + 3$, or a power or product of such primes, and γ is Euler's constant.” From (10) she concludes that $a_1/a < 0$, and therefore that Ramanujan's (3) is false.

Now there are two sign errors in (10). One should replace a_1/a by $-a_1/a$ and $\log 2/2$ by $-\log 2/2$. The first error probably came about by computing a_1/a as the logarithmic derivative of $f(s)\ s^{-2}\ \sqrt{s-1}$ for $s = 1$. But, from the definition of a_1 in (6), we have

$$f(s)\ s^{-2}\sqrt{s-1} = a + a_1(1-s) + \cdots,$$

and due to the change of sign, with $(1 - s)$ here, instead of the expected $(s - 1)$, this derivative is really $-a_1/a$. The second error was made in taking the generating function as

$$(11) \qquad f(s) = \left(1 - \frac{1}{2^s}\right)^{1/2}\left(\prod_q \frac{1}{1-q^{-2s}}\right)^{1/2}\{\zeta(s)L(s)\}^{1/2}$$

when it really is

$$(11') \qquad f(s) = \left(1 - \frac{1}{2^s}\right)^{-1/2}\left(\prod_q \frac{1}{1-q^{-2s}}\right)^{1/2}\{\zeta(s)L(s)\}^{1/2}.$$

Here $\zeta(s)$ and $L(s)$ are the well-known zeta and L functions:

$$\zeta(s) = \sum_1^\infty n^{-s}, \qquad L(s) = \sum_0^\infty (-1)^k(2k+1)^{-s}.$$

With (11) thus corrected, and rewriting the corrected (10) in a form more suitable to computation, we obtain from (8′) and (5) the following formula for c:

$$(12) \quad c = \frac{1}{2}\left(\frac{a_1}{a} - 1\right) = \frac{1}{2} + \frac{\log 2}{4} - \frac{\gamma}{4} - \frac{L'(1)}{4L(1)} - \frac{1}{4}\frac{d}{ds}\log\prod_q\left(\frac{1}{1-q^{-2s}}\right)\Bigg|_{s=1}.$$

3. Computation of b and c. The logarithmic derivative $L'(1)/L(1)$ may be expressed in terms of the so-called *lemniscate* constant, $\tilde{\omega}$, as follows:

$$(13) \qquad \frac{L'(1)}{L(1)} = \log\left[\left(\frac{\pi}{\tilde{\omega}}\right)^2\frac{e^\gamma}{2}\right].$$

This formula (or its equivalent) appears to have been discovered independently at least five times, by Berger [9], Lerch [10], de Séguier [11], Landau [12], and the author. Using (13), the first four terms on the right in (12) may be combined into

$$\frac{1}{2}\left[1 - \log\left(\frac{\pi}{\tilde{\omega}}\frac{e^{\gamma}}{2}\right)\right].$$

Since Gauss [13] computed $\log \tilde{\omega}$ to many places, and $\log \pi$, $\log 2$, and γ are well-known, we easily obtain

$$\frac{1}{2}\left[1 - \log\left(\frac{\pi}{\tilde{\omega}}\frac{e^{\gamma}}{2}\right)\right] = 0.4675804827 \tag{14}$$

for this combination.

The slowly convergent remaining term in (12), and the related product in (2), may be transformed by a technique of some general interest, since it is applicable to a whole class of related infinite products. For $s > \frac{1}{2}$ we may easily verify that

$$\left(\prod_q \frac{1}{1 - q^{-2s}}\right)^2 = \frac{\zeta(2s)(1 - 2^{-2s})}{L(2s)} \prod_q \frac{1}{1 - q^{-4s}}. \tag{15}$$

Hence, by recursion, we may transform (2) into the very rapidly converging product:

$$b = \frac{1}{\sqrt{2}} \prod_{k=1}^{\infty} \left\{ \frac{\zeta(2^k)(1 - 2^{-2^k})}{L(2^k)} \right\}^{(1/2)^{k+1}}. \tag{16}$$

From tables of $L(s)$ and $\zeta(s)\ (1 - 2^{-s})$, say in [14], we thus easily obtain

$$b = 0.764223654. \tag{17}$$

(A transformation similar to (15) is possible when q ranges over other arithmetic progressions [15].)

For the last term on the right side of (12) it is more convenient to apply the transformation (15) only twice. We obtain

$$\begin{aligned} \frac{d}{ds} \log \prod_q \frac{1}{1 - q^{-2s}}\bigg|_{s=1} &= \left(\frac{\zeta'(2)}{\zeta(2)} - \frac{L'(2)}{L(2)} + \frac{\log 2}{3}\right) \\ &+ \left(\frac{\zeta'(4)}{\zeta(4)} - \frac{L'(4)}{L(4)} + \frac{\log 2}{15}\right) - 2 \sum_q \frac{\log q}{q^8 - 1}. \end{aligned} \tag{18}$$

The last term here converges very rapidly to -0.0003356406. The quantities $\zeta'(n)/\zeta(n)$ have recently been computed by Rosser and Schoenfeld [16], but the corresponding logarithmic derivatives of $L(n)$ do not appear to be tabulated. The series

$$L'(n) = \frac{\log 3}{3^n} - \frac{\log 5}{5^n} + - \cdots \tag{19}$$

converge slowly for $n = 2$ and $n = 4$, but are of a type whose convergence may be accelerated by the $e_1{}^m$ nonlinear transformation [17]. Using this transformation on the partial sums of (19) we obtain

$$\frac{L'(2)}{L(2)} = 0.089065284 \quad \text{and} \quad \frac{L'(4)}{L(4)} = 0.011699896.$$

Then

$$\frac{d}{ds} \log \prod_q \frac{1}{1 - q^{-2s}} \bigg|_{s=1} = -0.457472706,$$

and finally

(20) $$c = 0.581948659.$$

4. A Table and Three Comparisons. From (5), (17), and (20) we have

(5′) $$B(x) = \frac{0.764223654\, x}{\sqrt{\log x}} \left[1 + \frac{0.581948659}{\log x} + O\left(\frac{1}{\log^2 x}\right) \right],$$

and since $c \neq \frac{1}{2}$, (3) is false. However, contrary to the remarks of Hardy and Stanley, since c is positive, and nearly $\frac{1}{2}$, we should expect Ramanujan's integral, $b \int_1^x (\log u)^{-1/2} du$, to approximate $B(x)$ much better than Landau's $bx(\log x)^{-1/2}$ does. In Table 1 we show that this is indeed the case.

In this table we tabulate $B(x)$ for $x = 2, 4, \cdots, 2^k, \cdots, 2^{26} = 67{,}108{,}864$. These counts were computed by Larry P. Schmid on an IBM 7090 [15]. We also tabulate Landau's function and the second-order approximation:

(21) $$l(x) = \frac{bx}{\sqrt{\log x}}, \qquad s(x) = \frac{bx}{\sqrt{\log x}} \left[1 + \frac{c}{\log x} \right].$$

TABLE 1

x	$B(x)$	$l(x)$	$r(x)$	$s(x)$	$B(x)/l(x)$	$B(x)/r(x)$	$B(x)/s(x)$
2	2	2	2	3	1.0894	1.2200	0.5922
2^2	3	3	3	4	1.1555	0.9583	0.8139
2^3	5	4	5	5	1.1793	0.9197	0.9214
2^4	9	7	9	9	1.2256	0.9635	1.0130
2^5	16	13	16	15	1.2180	0.9858	1.0429
2^6	29	24	29	27	1.2092	1.0103	1.0607
2^7	54	44	52	50	1.2160	1.0453	1.0858
2^8	97	83	94	92	1.1675	1.0273	1.0566
2^9	180	157	175	171	1.1490	1.0298	1.0510
2^{10}	337	297	327	322	1.1338	1.0310	1.0459
2^{11}	633	567	616	610	1.1168	1.0272	1.0376
2^{12}	1197	1085	1169	1161	1.1029	1.0237	1.0307
2^{13}	2280	2086	2231	2220	1.0932	1.0222	1.0269
2^{14}	4357	4019	4273	4260	1.0840	1.0196	1.0227
2^{15}	8363	7766	8215	8201	1.0768	1.0180	1.0198
2^{16}	16096	15039	15832	15828	1.0703	1.0167	1.0169
2^{17}	31064	29181	30628	30622	1.0645	1.0142	1.0144
2^{18}	60108	56717	59345	59362	1.0598	1.0129	1.0126
2^{19}	116555	110408	115208	115287	1.0557	1.0117	1.0110
2^{20}	226419	215225	224040	224260	1.0520	1.0106	1.0096
2^{21}	440616	420076	436343	436871	1.0489	1.0098	1.0086
2^{22}	858696	820836	850981	852161	1.0461	1.0091	1.0077
2^{23}	1675603	1605587	1661663	1664196	1.0436	1.0084	1.0069
2^{24}	3273643	3143562	3248231	3253531	1.0414	1.0078	1.0062
2^{25}	6402706	6160098	6356076	6366973	1.0394	1.0073	1.0056
2^{26}	12534812	12080946	12448925	12471056	1.0376	1.0069	1.0051

These are easily computed from (17) and (20) and are rounded to the nearest integer.

Ramanujan's function,

$$r(x) = b \int_1^x \frac{du}{\sqrt{\log u}}, \tag{22}$$

which was computed by the method indicated in the appendix below, was also rounded to the nearest integer.

Finally, Table 1 lists the ratios of $B(x)$ to the three approximation functions to 4D. All three of these functions underestimate $B(x)$. The results in Table 1 are consistent with the foregoing analysis. Ramanujan's $r(x)$ is a substantially better approximation than Landau's $l(x)$. But since

$$\frac{B(x)/l(x) - 1}{B(x)/r(x) - 1}$$

approaches a positive limit as $x \to \infty$, $r(x)$ has an error of the same order.

The error in $s(x)$ is about *twice* that of $r(x)$ for $x \approx 200$, and about equal to that of $r(x)$ for $x \approx 200{,}000$. Henceforth $s(x)$ is the best of the three. This temporary success of $r(x)$ is, of course, due to the fact that $s(x)$ ignores the third and higher order terms; and while these are surely not correctly represented by $r(x)$, the third term, at least, is of the correct sign.

In concluding this section we would point out the rather obvious fact that while

$$\frac{bx}{\sqrt{\log x}}\left[1 + \frac{c}{\log x} + \frac{d}{\log^2 x} + \frac{e}{\log^3 x} + \cdots\right]$$

is correct asymptotically, it is not very accurate for finite x. Two terms give us only $\frac{1}{2}$ % accuracy at $x \approx 70 \cdot 10^6$, and the higher coefficients, d, e, etc., can be calculated only with considerable labor. In contrast, $\int_2^x du/\log u$ agrees with $\pi(x)$ to about $\frac{1}{100}$% at $x \approx 70 \cdot 10^6$. An unsolved problem of interest is to find a replacement for the incorrect $r(x)$, that could be computed without undue difficulty by a *convergent* process, and which would be accurate to $O\left(\frac{x}{\log^m x}\right)$ for all m.

5. Odds and Evens and an Elementary Argument. The foregoing disproof of (3) is based on Landau's analysis, and this is based upon Cauchy's Theorem. It was certainly not available to Ramanujan in 1913, since, according to Hardy [3, p. 43], "he did not know Cauchy's Theorem." We raise the question whether we can give an elementary disproof of (3), i.e., one not based on Cauchy's Theorem. And we reply that there is a simple elementary argument which makes (3) highly unlikely (although it doesn't disprove it). Further, this argument arises in a very natural way as soon as we begin to compute $B(x)$.

An even number $2n$ is expressible in the form $u^2 + v^2$ if and only if n is, since

$$2n = u^2 + v^2 \rightleftarrows n = \left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2. \tag{23}$$

Consider

$$B(x) = O_1(x) + E_1(x) \tag{24}$$

where $O_1(x)$ counts the odd numbers of this form and $E_1(x)$ counts the even numbers. In view of (23) we thus have

$$E_1(x) = B\left(\frac{x}{2}\right). \tag{25}$$

Hence, to compute $B(x)$, it suffices to compute $O_1(x)$, and to obtain $E_1(x)$ and $B(x)$ by the recursions:

$$\begin{aligned} E_1(2x) &= B(x), \\ B(2x) &= E_1(2x) + O_1(2x). \end{aligned} \tag{26}$$

Since

$$O_1(x) \sim \tfrac{1}{2} B(x), \tag{27}$$

as we shall see, this means a saving in computation of 50%. (This is, in fact, the way in which the $B(x)$ of Table 1 was computed. See Table 2.)

The generating function $f(s)$ for $B(x)$, given above by (11′), may also be written

$$f(s) = \frac{1}{1 - 2^{-s}} \prod_p \frac{1}{1 - p^{-s}} \prod_q \frac{1}{1 - q^{-2s}} \tag{28}$$

where the p's are the primes of the form $4m + 1$ and the q's are the primes of the form $4m + 3$. Correspondingly, that for $O_1(x)$ is the very similar

$$f'(s) = \prod_p \frac{1}{1 - p^{-s}} \prod_q \frac{1}{1 - q^{-2s}}, \tag{29}$$

and just as (28) leads to

$$B(x) \sim \frac{bx}{\sqrt{\log x}},$$

so (29) leads to

$$O_1(x) \sim \frac{\frac{1}{2}bx}{\sqrt{\log x}}.$$

Now, by whatever (fallacious) reasoning Ramanujan obtained (3) from (28), it seems likely that he would have similarly obtained

$$O_1(x) = \frac{1}{2} K \int_1^x \frac{du}{\sqrt{\log u}} + O\left(\frac{x}{\log x}\right)^{1/2} \tag{30}$$

from (29). (This would again be in analogy with prime number theory, since the number of primes in the arithmetic progressions $4m + 1$ or $4m + 3$, say, are both given by $\frac{1}{2} \int_2^x du/\log u + O\left(\frac{x}{\log^m x}\right)$ for any m). But (3) and (30) quickly lead to a contradiction. Assume that

$$(31) \qquad O_1(x) = \frac{b'x}{\sqrt{\log x}}\left[1 + \frac{c'}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

Together with

$$(32) \qquad B(x) = \frac{bx}{\sqrt{\log x}}\left[1 + \frac{c}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right]$$

and (24) we obtain

$$(33) \qquad E_1(x) = \frac{(b - b')x}{\sqrt{\log x}} + \frac{(bc - b'c')x}{(\log x)^{3/2}} + O\left(\frac{x}{\log^{5/2} x}\right).$$

But from (25) and (32) we also have

$$(34) \qquad E_1(x) = \frac{\frac{1}{2}bx}{\sqrt{\log x - \log 2}}\left[1 + \frac{c}{\log x - \log 2} + O\left(\frac{1}{\log^2 x}\right)\right],$$

and since

$$(34a) \qquad \frac{1}{\sqrt{\log x - \log 2}} = \frac{1}{\sqrt{\log x}}\left[1 + \frac{\frac{1}{2}\log 2}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right],$$

by comparing (33) and (34) we obtain

$$(35) \qquad b' = \tfrac{1}{2}b$$

and

$$(36) \qquad c' = c - \tfrac{1}{2}\log 2.$$

Now (35) is consistent with (27). But (36) indicates that c and c' cannot *both* equal $\frac{1}{2}$. Therefore at least one of (3) and (30) must be false. But the generating functions $f(s)$ in (28) and $f'(s)$ in (29) are very similar. Neither could be said to be more "fundamental" in any reasonable sense. There is no more reason for (3) to be true than for (30) to be true. By the Principle of Sufficient Reason it would be most likely, therefore, if neither were true. And this, as we now know, is the case.

Carrying out the analysis of section 2 with the generating function $f'(s)$ we obtain and record

THEOREM 1.

$$(37) \qquad O_1(x) = \frac{\frac{1}{2}bx}{\sqrt{\log x}}\left[1 + \frac{c'}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right]$$

and

$$(38) \qquad E_1(x) = \frac{\frac{1}{2}bx}{\sqrt{\log x}}\left[1 + \frac{c''}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right]$$

where

$$(39) \qquad \begin{aligned} \tfrac{1}{2}b &= 0.382111827, \\ c' &= c - \tfrac{1}{2}\log 2 = 0.235375069, \\ c'' &= c + \tfrac{1}{2}\log 2 = 0.928522249. \end{aligned}$$

It follows that

$$(40) \qquad E_1(x) - O_1(x) \sim \frac{1}{\log_2 x} E_1(x).$$

(The fact that the simple equation (40) is free of the constants b and c is suggestive of the existence of an elementary theory such as we have discussed above.)

In contrast with the differing second-order coefficients in (39), consider now a different partition of the integers $n = u^2 + v^2$. Let $B_4(x)$ be the number of integers $\leqq x$ of the form $u^2 + 4v^2$. These integers constitute the subset of the integers $n = u^2 + v^2$ for which $n = 4m$ or $n = 4m + 1$. Correspondingly, $B(x) - B_4(x)$ counts those of the form $4m + 2$. We then have

THEOREM 2.

$$(41) \qquad B_4(x) = \frac{\frac{3}{4}bx}{\sqrt{\log x}}\left[1 + \frac{c}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

$$(42) \qquad B(x) - B_4(x) = \frac{\frac{1}{4}bx}{\sqrt{\log x}}\left[1 + \frac{c}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

Proof. The integers counted by $B(x) - B_4(x)$ are those of the form

$$n = (2u + 1)^2 + (2v + 1)^2 = 4m + 2.$$

For such an n,

$$n/2 = (u - v)^2 + (u + v + 1)^2 = 2m + 1.$$

Hence,

$$(43) \qquad B(x) - B_4(x) = O_1\left(\frac{x}{2}\right),$$

and from (37) and (34a) we thus obtain (42). Then from (5) we obtain (41).

By elementary means—that is, by algebraic and arithmetic calculations, but no new analysis—the reader may obtain, if he wishes, the following results which are more precise than those of equations (40) and (41).

$$(40a) \qquad O_1(x) = E_1(x)\left[1 - \frac{1}{\log_2 x} - \frac{1.92914889}{(\log_2 x)^2} - O(\log_2 x)^{-3}\right].$$

$$(41a) \qquad B_4(x) = \tfrac{3}{4} B(x)\left[1 + \frac{0.25}{(\log_2 x)^2} + \frac{1.46457444}{(\log_2 x)^3} + O(\log_2 x)^{-4}\right].$$

Let us also consider the subset of the integers $n = u^2 + 4v^2$ consisting of those for which the largest power of 2 dividing n is an even power. That is, n equals 4^k times an odd number for $k = 0, 1, 2 \cdots$. Let $B_4'(x)$ be the number of such integers $\leqq x$. Now we have the generating function:

$$(44) \qquad f_4'(s) = \frac{1}{1 - 2^{-2s}} \prod_p \frac{1}{1 - p^{-s}} \prod_q \frac{1}{1 - q^{-2s}},$$

and from this we derive

THEOREM 3.

$$B_4'(x) = \frac{b_4' x}{\sqrt{\log x}}\left[1 + \frac{c_4'}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right] \tag{45}$$

where

$$b_4' = \tfrac{2}{3} b = 0.509482436 \tag{46}$$

and

$$c_4' = c - \tfrac{1}{6}\log 2 = 0.466424129. \tag{47}$$

Alternatively, we may again use such elementary computations as those in equations (31) — (36), this time using the relations

$$\begin{aligned} O_4'(x) &= O_1(x) \\ B_4'(x) &= O_4'(x) + E_4'(x) \\ E_4'(x) &= B_4'(x/4). \end{aligned} \tag{48}$$

Similarly, we may compute $E_4'(x)$ and $B_4'(x)$ by recursion from $O_4'(x) = O_1(x)$. This is done in Table 2. Since c_4', which equals 0.466424129, is even closer to $\frac{1}{2}$ than c is, we might expect $\frac{2}{3}\, r(x)$ to be a good approximation for $B_4'(x)$. It is, in

TABLE 2

x	$O_1(x) = O_4'(x)$	$E_1(x)$	$E_4'(x)$	$B_4'(x)$	$\frac{2}{3}r(x)$	$B_4'(x)/\frac{2}{3}r(x)$
2	1	1	0	1	1	0.9150
2^2	1	2	1	2	2	0.9583
2^3	2	3	1	3	4	0.8278
2^4	4	5	2	6	6	0.9635
2^5	7	9	3	10	11	0.9242
2^6	13	16	6	19	19	0.9929
2^7	25	29	10	35	34	1.0162
2^8	43	54	19	62	63	0.9849
2^9	83	97	35	118	117	1.0127
2^{10}	157	180	62	219	218	1.0050
2^{11}	296	337	118	414	411	1.0077
2^{12}	564	633	219	783	780	1.0044
2^{13}	1083	1197	414	1497	1487	1.0067
2^{14}	2077	2280	783	2860	2849	1.0039
2^{15}	4006	4357	1497	5503	5477	1.0048
2^{16}	7733	8363	2860	10593	10555	1.0036
2^{17}	14968	16096	5503	20471	20419	1.0026
2^{18}	29044	31064	10593	39637	39563	1.0019
2^{19}	56447	60108	20471	76918	76805	1.0015
2^{20}	109864	116555	39637	149501	149360	1.0009
2^{21}	214197	226419	76918	291115	290896	1.0008
2^{22}	418080	440616	149501	567581	567321	1.0005
2^{23}	816907	858696	291115	1108022	1107775	1.0002
2^{24}	1598040	1675603	567581	2165621	2165487	1.0001
2^{25}	3129063	3273643	1108022	4237085	4237384	0.9999
2^{26}	6132106	6402706	2165621	8297727	8299283	0.9998

fact, better than one would expect. Presumably the small errors in the second term are partially compensated for by small errors of opposite sign in the higher terms.

Finally, we might mention the general problem, $B_n(x)$, for numbers of the form $u^2 + nv^2$. For indefinite forms, $n < 0$, and for such cases as $n = 6$, where the so-called class number exceeds unity, there are interesting complications. These will be discussed in a forthcoming paper, [15].

Appendix (The computation of $r(x)$).

Ramanujan's function $r(x)$, which is given by (22), may be transformed into

$$r(x) = 2bx\left\{\frac{1}{x}\int_0^{\sqrt{\log x}} e^{v^2}\,dv\right\}$$

by $u = e^{v^2}$. With $\sqrt{\log x} = w$ the bracket becomes Dawson's integral:

$$F(w) = e^{-w^2}\int_0^w e^{v^2}\,dv.$$

Rosser [18] has given 10D values of $F(w)$ for selected arguments w. He recommends Lagrange interpolation for intermediate arguments, but more accurate

TABLE 3

k	$\int_1^{2^k} \frac{du}{\sqrt{\log u}}$
1	2.14503760
2	4.09644933
3	7.11347310
4	12.2226993
5	21.2384587
6	37.5592045
7	67.6003252
8	123.556490
9	228.714206
10	427.706202
11	806.349552
12	1530.09977
13	2918.71994
14	5591.49845
15	10750.0708
16	20716.8362
17	40077.2671
18	77653.3419
19	150751.822
20	293160.823
21	570962.936
22	1113523.90
23	2174314.55
24	4250366.94
25	8317036.10
26	16289636.0
27	31931697.5

(and more interesting) computations utilize the Taylor series based on the nearest w which he tabulated. This is possible since $F(w)$ satisfies a first-order differential equation:

$$F'(w) = -2\,wF(w) + 1.$$

Thus, the coefficients in the Taylor series, $c_n(w) = \dfrac{1}{n!}\dfrac{d^nF(w)}{dw^n}$, may be readily obtained by recursion from $F(w)$ [18, p. 179]:

$$c_{n+2}(w) = -\frac{2}{n+2}\{wc_{n+1}(w) + c_n(w)\}.$$

In Table 3 we tabulate $\int_1^{2^k} (\log u)^{-1/2}\,du$ to 9 significant figures for future reference. From these values $r(x)$ is obtained by (22).

Applied Mathematics Laboratory
David Taylor Model Basin
Washington 7, D. C.

1. Edmund Landau, "Uber die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindeszahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate," *Archiv der Math. und Physik* (3), v. 13, 1908, p. 305–312.

2. Edmund Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, v. II, Chelsea, N. Y., 1953, p. 643–645, 668–669.

3. G. H. Hardy, *Ramanujan*, Chelsea, N. Y., 1959, p. 61–62.

4. W. J. LeVeque, *Topics in Number Theory*, v. II., Addison-Wesley, Reading, 1956, p. 257–263.

5. G. H. Hardy, Editor, *Collected Papers of Srinivasa Ramanujan*, Cambridge, 1927, p. xxiv, xxviii.

6. G. K. Stanley, "Two assertions made by Ramanujan," *J. London Math. Soc.*, v. 3, 1928, p. 232–237.

7. G. K. Stanley, "Corrigenda," *J. London Math. Soc.*, v. 4, 1929, p. 32.

8. Daniel Shanks, "The second-order term in the asymptotic expansion of $B(x)$," Abstract 599–46, *Notices*, American Math. Soc., v. 10, 1963, p. 261. Errata, p. 377.

9. A. Berger, "Sur une sommation de quelque séries," *Nova acta regiae Soc. Sc. Upsaliensis* (3), v. 12, 1883, p. 30.

10. M. Lerch, "Sur quelques formules relative au nombre des classes," *Bull. des sc. Math.* (2), v. 21, 1897, p. 302–303.

11. J. de Séguier, "Sur certaines sommes arithmétiques," *Jour. de math. pure appl.* (5), v. 5, 1899, p. 55, 77.

12. Edmund Landau, "Ueber die zu einem algebraischen zahlkörper gehörige Zetafunction und die Ausdehnung der Tschebyschefschen Primzahlentheorie auf das Problem der Vertheilung der Primideale," *J. reine Angew. Math.*, v. 125, 1903, p. 134–136, 176–179.

13. C. F. Gauss, "De curva lemniscata," *Werke*, v. 3, Göttingen, 1876, p. 414. (The last 6 of the 25 digits here have been corrected in John W. Wrench, Jr., MTE 132, *MTAC*, v. 3, 1948, p. 202.)

14. Daniel Shanks & John W. Wrench, Jr., "The calculation of certain Dirichlet series," *Math. Comp.*, v. 17, 1963, p. 138, 151.

15. Daniel Shanks & Larry P. Schmid, "Variations on a theorem of Landau," (to appear).

16. J. Barkley Rosser & Lowell Schoenfeld, "Approximate formulas for some functions of prime numbers," *Illinois Jour. of Math.*, v. 6, 1962, p. 93.

17. Daniel Shanks, "Non-linear transformations of divergent and slowly convergent sequences," *Jour. Math. Phys.*, v. 34, 1955, p. 1–42.

18. J. Barkley Rosser, *Theory and Application of* $\int_0^z e^{-x^2}\,dx$ *and* $\int_0^z e^{-p^2y^2}\,dy \int_0^y e^{-x^2}\,dx$, Mapleton House, Brooklyn, 1948, p. 190–191.