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Polylogarithms, Dirichlet Series, and Certain Constants

By Daniel Shanks

The *polylogarithms* $F_s(z)$ are defined by

$$F_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s} \tag{1}$$

for $|z| < 1$ and for the real part of $s \geqq 0$, and by analytic continuation for other values of z and s. They can be regarded as functions of z, with a parameter s, given by the power series (1), or as functions of s, with a parameter z, given by the Dirichlet series (1).

Recently [1] we discussed the Dirichlet series defined by

$$L_a(s) = \sum_{k=0}^{\infty} \frac{\left(\frac{-a}{2k+1}\right)}{(2k+1)^s} \tag{2}$$

and its analytic continuation, where $\left(\frac{-a}{2k+1}\right)$ is the Jacobi symbol. It is expressible in closed form for three-quarters of all combinations of integers a and s; namely, for $s \leqq 1$ and all a, for s even and >1 if $a < 0$, and for s odd and > 1 if $a > 0$.

The remaining, non-closed form $L_a(n)$ for $a = \pm 2, \pm 3$, and ± 6, with $n \leqq 10$, were computed [1] by a device, which (in essence) is based on the fact that all of the so-called *characters* modulo 8, 12, or 24 are real. In contrast, the corresponding $L_a(n)$ for $a = \pm 5, \pm 7$, and ± 10, say, which were also desired, are not obtainable by that method, unless it is modified, since now some of the characters are complex.

We did, however, express $L_a(s)$ as a linear combination of the functions $S_s(x)$ or $C_s(x)$ for various values of x determined by the integer a [1, equations (24)–(27)]. These functions [1, equation (18)] are defined by

$$\begin{aligned} S_s(x) &= \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^s}, \\ C_s(x) &= \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^s}. \end{aligned} \tag{3}$$

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Now consider the real and imaginary parts of $F_s(e^{i\pi\alpha/2})$. We will call them

$$(4) \qquad \begin{aligned} R_s(\alpha) &= \mathcal{R} F_s(e^{i\pi\alpha/2}) \\ I_s(\alpha) &= \mathcal{I} F_s(e^{i\pi\alpha/2}). \end{aligned}$$

It follows that

$$(5) \qquad \begin{aligned} C_s(x) &= R_s(4x) - \frac{1}{2^s} R_s(8x) \\ S_s(x) &= I_s(4x) - \frac{1}{2^s} I_s(8x). \end{aligned}$$

By the aforementioned linear combinations we may, therefore, express $L_a(s)$ in terms of the special polylogarithms (4). For example, we have

$$L_5(s) = \frac{2}{\sqrt{5}}\left[I_s(0.2) - \frac{1}{2^s} I_s(0.4) + I_s(0.6) - \frac{1}{2^s} I_s(1.2)\right],$$

$$L_{-5}(s) = \frac{2}{\sqrt{5}}\left(1 + \frac{1}{2^s}\right)[R_s(0.8) - R_s(1.6)],$$

$$(6) \qquad \begin{aligned} L_{10}(s) = \frac{2}{\sqrt{10}}\Big[I_s(0.1) &- \frac{1}{2^s} I_s(0.2) - I_s(0.3) + \frac{1}{2^s} I_s(0.6) \\ &+ I_s(0.7) + I_s(0.9) - \frac{1}{2^s} I_s(1.4) - \frac{1}{2^s} I_s(1.8)\Big], \end{aligned}$$

$$\begin{aligned} L_{-10}(s) = \frac{2}{\sqrt{10}}\Big[R_s(0.1) &- \frac{1}{2^s} R_s(0.2) + R_s(0.3) - \frac{1}{2^s} R_s(0.6) \\ &- R_s(0.7) + R_s(0.9) + \frac{1}{2^s} R_s(1.4) - \frac{1}{2^s} R_s(1.8)\Big]. \end{aligned}$$

The Computation Staff of the Amsterdam Mathematisch Centrum, under the direction of Dr. A. van Wijngaarden, has computed [2] several tables of polylogarithms accurate to 10D. Their Table III gives $R_s(\alpha)$ and $I_s(\alpha)$ for $s = 1(1)12$ and $\alpha = 0(0.01)2$. The numbers on the right side of (6) for integral s are therefore given explicitly in this table, and thus, with some simple arithmetic, we obtain our Table 1.

TABLE 1

s	$L_5(s)$	$L_{-5}(s)$	$L_{10}(s)$	$L_{-10}(s)$
1	1.404962946	0.6456134114	0.9934588266	1.150086523
2	1.128043325	0.8827642541	0.9314284985	1.092365033
3	1.039982136	0.9616778624	0.9682482537	1.034721928
4	1.012801468	0.9874205162	0.9883161275	1.012021984
5	1.004182100	0.9958455012	0.9959695576	1.004067704
6	1.001381310	0.9986219811	0.9986393802	1.001364688
7	1.000458601	0.9995417817	0.9995442414	1.000456202
8	1.000152606	0.9998474373	0.9998477867	1.000152262
9	1.000050832	0.9999491729	0.9999492226	1.000050783
10	1.000016939	0.9999830616	0.9999830687	1.000016932

TABLE 2
The Hardy-Littlewood Constants

$h_{-10} = 0.67111392$	$h_{-3} = 1.38342429$	$h_4 = 1.37281346$
$h_{-9} = 0$	$h_{-2} = 1.85005441$	$h_5 = 0.52824557$
$h_{-8} = 1.85005441$	$h_{-1} = 0$	$h_6 = 0.71304162$
$h_{-7} = 0.75737123$	$h_0 = 0$	$h_7 = 1.97304317$
$h_{-6} = 1.03575587$	$h_1 = 1.37281346$	$h_8 = 0.71306310$
$h_{-5} = 1.77330507$	$h_2 = 0.71306310$	$h_9 = 0.91520897$
$h_{-4} = 0$	$h_3 = 1.12073275$	$h_{10} = 1.08240211$

From Table 1, in turn, we may compute [3], [4] the Hardy-Littlewood constants h_a for $a = \pm 5$ and ± 10. Together with previously computed values, we may thus complete an 8D table of h_a for $a = -10(1)10$ except for $a = \pm 7$. The $L_{\pm 7}(s)$, needed to fill this gap, may also be expressed in terms of $I_s(\alpha)$ and $R_s(\alpha)$, but this time the arguments α are not given explicitly in [2], and elaborate interpolation would be required to obtain comparable precision.

Alternatively, as is known, generalized harmonic series, including $L_a(s)$ for integer s, may be expressed in terms of the *polygamma* functions [5], [6]. However, the same difficulty arises for $L_{\pm 7}(s)$, and again elaborate and laborious interpolation is necessary. At the author's request John W. Wrench, Jr. has kindly computed $L_7(2)$, $L_7(4)$, $L_{-7}(3)$ and $L_{-7}(5)$ in this way, and these numbers, together with the closed-form $L_{\pm 7}(s)$, suffice to complete our tabulation of h_a. This is given in Table 2.

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New Factors of Fermat Numbers

By Claude P. Wrathall

Eleven new factors of Fermat numbers $F_m = 2^{2^m} + 1$ are listed below. A summary of the present status of the sequence F_m is presented in Table 2.

The method used was suggested by Dr. J. L. Selfridge. Simply stated, the method consisted of forming a sieve array to eliminate possible factors divisible by a prime $\leqq 499$. The remaining possible factors were tested to determine if any of the congruence relationships

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