

On the Stability of Parabolic Difference Schemes

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1. **Introduction.** Consider a differential operator of order $2m$

$$P(x, \partial) = \sum_{|\nu| \leq 2m} A_\nu(x) \partial_1^{\nu_1} \cdots \partial_s^{\nu_s},$$

where $\nu = (\nu_1, \dots, \nu_s)$, $|\nu| = \sum \nu_i$ and $\partial_j = \partial/\partial x_j$. $A_\nu(x)$ are complex valued, uniformly bounded $n \times n$ matrices depending on $x = (x_1, \dots, x_s)$, $-\infty < x_i < \infty$, in a real vector space R_s . $I \cdot \partial/\partial t - P(x, \partial)$, where I is the identity matrix, is said to be uniformly parabolic if there exists a constant $\delta > 0$ such that all roots σ_μ of

$$(1.1) \quad \det \left\{ \sum_{|\nu|=2m} A_\nu(x) (i\omega_1)^{\nu_1} \cdots (i\omega_s)^{\nu_s} - \sigma I \right\} = 0$$

satisfy

$$(1.2) \quad \operatorname{Re} \sigma_\mu \leq -\delta$$

for all $x \in R_s$ and all $|\omega| = 1$.¹ Here $\omega \in R_s$ is a dual variable.

In this paper we shall study the stability of difference approximations to the initial value problem:

$$(1.3) \quad \begin{aligned} \partial u/\partial t &= P(x, \partial)u, & 0 \leq t \leq T < \infty, \\ u(x, 0) &= f(x), \end{aligned}$$

where $u(x, t) = (u_1(x, t), \dots, u_n(x, t))'$ is a sufficiently smooth vector which for each fixed value of t belongs to a complex Hilbert space S_0 with an inner product and a norm defined by

$$(u, v)_0 = \int_{R_s} u^* \cdot v \, dx,$$

$$\|u\|_0^2 = (u, u)_0.$$

In the following we shall always suppose that (1.3) is a properly posed initial value problem in the sense of Richtmyer [17],² and only engage in the study of stability conditions for difference approximations. The existence, uniqueness, etc., of solutions of parabolic systems of differential equations have been studied by, among others, Aronson [1], [2], Eidel'man [4], [5], [6] and Mizohata [15], [16]. We refer the reader to these papers for such questions.

Received July 6, 1964.

¹ By y' and y^* we denote the transpose and the conjugate transpose of a vector y . Similar notations hold for a matrix A . By $|y|$ and $|A| = \sup_{|y|=1} |Ay|$ we denote the euclidean norm of y and A .

² For the basic theory on difference approximations to initial value problems we refer the reader to this book.

The difference schemes to be studied are of the multistep type:

$$(1.4) \quad u(x, (n + 1)k, h) = \sum_{\mu=0}^r \alpha_\mu u(x, (n - \mu)k, h) + \sum_{\mu=-1}^r Q_\mu(x, D, h)u(x, (n - \mu)k, h),$$

$$r \leq n \leq T/k - 1; T/k = \text{integer}.$$

We suppose that $u(x, \nu k, h)$ are given for $\nu = 0, 1, \dots, r - 1$, and that $u(x, 0, h) = f(x)$ where $f(x)$ is defined in (1.3). α_μ are real constants. k is the time step which is connected with the space width h by an equation $k = \lambda h^{2m}$, $\lambda = \text{constant} > 0$. $Q_\mu(x, D, h)$ are polynomials in the variables $h, hD_{0j}(h)$ and $hD_{\pm j}(h)$ with uniformly bounded matrix coefficients. The terms of these polynomials can be written $B_\mu^{\sigma\tau}(x)h^\sigma(hD)^\tau$ with $\sigma + \tau \geq 2m$. Here $(hD)^\tau = (hD) \dots (hD)$, where each hD stands for one of the operators $hD_{0j}(h), hD_{+j}(h), hD_{-j}(h)$. These difference operators are defined by

$$2hD_{0j}(h)u(x) = u(x + he_j) - u(x - he_j),$$

$$hD_{\pm j}(h)u(x) = \pm(u(x \pm he_j) - u(x)),$$

where e_j is the unit vector in the direction of the j th coordinate axis. The elements of $B_\mu^{\sigma\tau}(x)$ are supposed to have sufficiently many uniformly bounded derivatives. The sum of the terms for which $\sigma = 0$ and $\tau = 2m$ are denoted $Q_\mu^{(1)}(x, D, h)$, respectively.

Following Richtmyer [17] we introduce a new inner product space S_r . The elements of S_r are $(r + 1)$ -component vectors, whose components are elements in S_0 . The inner product and the norm are defined by

$$(1.5) \quad (\tilde{u}, \tilde{v})_r = \sum_{i=0}^r (u_i, v_i)_0,$$

$$\|\tilde{u}\|_r^2 = (\tilde{u}, \tilde{u})_r,$$

$$\tilde{u} = (u_0, u_1, \dots, u_r)',$$

$$\tilde{v} = (v_0, v_1, \dots, v_r)', \quad u_i \in S_0, v_i \in S_0, i = 0, 1, \dots, r.$$

We shall always suppose that $(I - Q_{-1}(x, D, h))^{-1}$ and $(I - Q_{-1}(x_0, D, h))^{-1}$, $x_0 \in R_s$, exist and that they are defined for all $u \in S_0$. We also suppose that these operators are uniformly bounded, that is, the bound for $\|(I - Q_{-1}(x, D, h))^{-1}\|_0$ is independent of h and the bound for $\|(I - Q_{-1}(x_0, D, h))^{-1}\|_0$ is independent of x_0 and h . We are then able to transform (1.4) into an explicit one-step formula:

$$(1.6) \quad \tilde{u}(x, (n + 1)k, h) = \tilde{Q}\tilde{u}(x, nk, h),$$

$$\tilde{Q} = \begin{pmatrix} (I - Q_{-1})^{-1}(\alpha_0 I + Q_0) & \dots & (I - Q_{-1})^{-1}(\alpha_r I + Q_r) \\ I & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & I & \dots & 0 \end{pmatrix},$$

$$\tilde{u}(x, nk, h) = (u(x, nk, h), \dots, u(x, (n - r)k, h))'.$$

As in the case of differential equations many properties of difference schemes are stated in terms of characteristic matrices. We therefore, denoting by

$$\xi_j = \omega_j h, \quad j = 1, \dots, s,$$

define the matrices $\hat{Q}_\mu(x, \xi, h)$, $\hat{Q}_\mu^{(1)}(x, \xi, h)$ and $\hat{Q}(x, \xi, h)$ which are constructed by exchanging $hD_{0j}(h)$ and $hD_{\pm j}(h)$ for $i \cdot \sin \xi_j$ and $2i \cdot \sin(1/2)\xi_j \cdot e^{\pm i(1/2)\xi_j}$, respectively, in $Q_\mu(x, D, h)$, $Q_\mu^{(1)}(x, D, h)$ and $\hat{Q}(x, D, h)$. In the case when Q_μ do not depend on x this amounts to the same thing as taking the Fourier transforms for

$$\begin{aligned} (hD_{0j}(h)f(x))^\wedge &= i \cdot \sin \xi_j \cdot \hat{f}, \\ (hD_{\pm j}(h)f(x))^\wedge &= 2i \cdot \sin(1/2)\xi_j \cdot e^{\pm i(1/2)\xi_j} \cdot \hat{f}, \\ \hat{f} &= \text{Fourier transform of } f(x). \end{aligned}$$

It follows immediately that \hat{Q}_μ , $\hat{Q}_\mu^{(1)}$ and \hat{Q} are 2π -periodic matrix-valued functions of $\xi_j, j = 1, \dots, s$. Observe especially that $\hat{Q}_\mu^{(1)}$ depend only on ξ and x and that $\hat{Q}_\mu^{(1)} = O(|\xi|^{2m})$.

We write \hat{Q} as the sum of three matrices:

$$\hat{Q} = \hat{Q}^{(0)} + \hat{Q}^{(1)} + \hat{Q}^{(2)},$$

where

$$\hat{Q}^{(0)} = \begin{pmatrix} \alpha_0 I & \alpha_1 I & \dots & \alpha_{r-1} I & \alpha_r I \\ I & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & I & 0 \end{pmatrix}$$

and

$$\hat{Q}^{(0)} + \hat{Q}^{(1)} = \begin{pmatrix} (I - \hat{Q}_{-1}^{(1)})^{-1}(\alpha_0 I + \hat{Q}_0^{(1)}) & \dots & (I - \hat{Q}_{-1}^{(1)})^{-1}(\alpha_r I + \hat{Q}_r^{(1)}) \\ I & \dots & 0 \\ & \dots & \\ 0 & \dots & I & 0 \end{pmatrix}.$$

Observe that the existence of $(I - \hat{Q}_{-1}^{(1)})^{-1}$ follows from the existence of $(I - \hat{Q}_{-1})^{-1}$, the fact that $\hat{Q}_{-1}^{(1)}$ depends on ξ and x only and the fact that $\hat{Q}_{-1} - \hat{Q}_{-1}^{(1)} = O(h)$. We also introduce the $(r + 1) \times (r + 1)$ matrix

$$(1.7) \quad \alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_r \\ 1 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

The stability conditions of this paper are stated in terms of $\kappa_i, |\kappa_j| \leq |\kappa_i|, 1 \leq i < j \leq (r + 1)n$, the eigenvalues of $\hat{Q}^{(0)} + \hat{Q}^{(1)}$ and $\gamma_i, |\gamma_j| \leq |\gamma_i|, 1 \leq i < j \leq r + 1$, the eigenvalues of α . This is very convenient from a practical point of view. Observe that κ_i can be calculated as the roots of

$$(1.8) \quad \det \{ \kappa^{r+1}(I - \hat{Q}_{-1}^{(1)}) - \kappa^r(\alpha_0 I + \hat{Q}_0^{(1)}) - \dots - (\alpha_r I + \hat{Q}_r^{(1)}) \} = 0.$$

Following Richtmyer [17] we define:

Definition 1. (1.4) is a consistent difference approximation to (1.3) if for all sufficiently smooth solutions $u(x, t)$ to $(I \cdot \partial/\partial t - P(x, \partial))u = 0$:

$$\left\| (I - Q_{-1})u(x, t + k) - \sum_{\mu=0}^r (\alpha_\mu I + Q_\mu)u(x, t - \mu k) \right\|_0 = o(k).$$

Using Taylor's theorem we immediately obtain that if (1.4) is a consistent difference approximation:

$$(1.9) \quad \sum_{\mu=0}^r \alpha_{\mu} = 1,$$

$$(1.10) \quad \left(1 + \sum_{\mu=0}^r \mu \alpha_{\mu}\right) kP(x, \partial) = \sum_{\mu=-1}^r Q_{\mu} + o(k).$$

Because of the fact that various stability definitions are used in the literature, we here give two definitions.

Definition 2. (1.6) is a stable difference scheme if there exists a constant C_1 independent of h such that for all $\tilde{u}_r \in S_r$

$$(1.11) \quad \|\tilde{u}_n\|_r \leq C_1 \|\tilde{u}_r\|_r$$

for all $n \leq T/k$, $\tilde{u}_n = (u(x, mk, h), \dots, u(x, (m-r)k, h))'$.

Definition 3. (1.6) is a strongly stable difference scheme if there exists a constant C_2 , independent of h , and a norm $\|\cdot\|_{\text{eq}}$, equivalent to $\|\cdot\|_r$, such that

$$(1.12) \quad \|\tilde{u}_{n+1}\|_{\text{eq}} \leq (1 + C_2 k) \|\tilde{u}_n\|_{\text{eq}}$$

for all $n < T/k$.

(1.12) is easily seen to be a condition of at least the same strength as (1.11). The significance of strong stability is thoroughly discussed in a paper by Kreiss [12].

In analogy to (1.1) and (1.2) we define:

Definition 4. (1.6) is a parabolic difference scheme if there exists a constant $\delta_1 > 0$ such that for all $|\xi_j| \leq \pi$ and all $x \in R_s$

$$(1.13) \quad |\kappa_i| \leq 1 - \delta_1 |\xi|^{2m}, \quad i = 1, \dots, (r+1)n.$$

Remark. If we choose $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = \dots = \alpha_r = 0$ and

$$Q_0 = \lambda \sum_{|\nu| \leq 2m} A_{\nu}(x) h^{2m-|\nu|} (hD_{01}(h))^{\nu_1} \dots (hD_{0s}(h))^{\nu_s},$$

$$Q_{\mu} = 0, \quad \mu \geq 1, \quad Q_{-1} = 0,$$

we get, as is easily verified using (1.2), a consistent parabolic difference approximation to (1.3) provided λ is chosen sufficiently small. This shows that there always exists a consistent parabolic difference scheme to each uniformly parabolic differential operator.

We are now able to state our results.

THEOREM 1. Let (1.4) be a consistent difference approximation to (1.3). Under the assumptions

- (a) $I \cdot \partial / \partial t - P(x, \partial)$ is uniformly parabolic,
- (b) the eigenvalues γ_i , $i = 2, \dots, r+1$, lie in the interior of the unit circle,
- (c) for every $\xi \neq 0$, all eigenvalues κ_i of $\hat{Q}^{(0)} + \hat{Q}^{(1)}$ lie in the interior of the unit circle,

the difference scheme is parabolic.

THEOREM 2. Suppose that (1.6) is a parabolic difference scheme and that all eigenvalues γ_i of \mathfrak{A} with modulus = 1 are simple. Then it is strongly stable.

Observe that in Theorem 2 we are able to weaken the conditions on γ_i compared with those in Theorem 1.

In Section 2 we prove Theorem 1. In Section 3 we prove, using results by Kreiss [12], [13], a fundamental inequality, which essentially corresponds to Theorem 2 in the case when \tilde{Q} has constant coefficients. In Section 4 we construct a new norm and state some lemmas. Using the results of Section 3 and Section 4 we are able to give a proof of Theorem 2 in Section 5. Here we make use of a well-known device due to Gårding [7].

For convenience we only treat homogeneous equations with coefficients independent of t . The extension of the results of this paper to the general case, however, presents no new difficulties.

The techniques used in this paper are closely related to those used by Lax and Wendroff [14] and Kreiss [10]–[13]. A slightly different approach to difference approximations to parabolic differential equations is found in papers by John [9] and Aronson [3]. Using a similar type of condition as we do, John [9], working with a maximum norm, proved the stability of difference approximations to a single second order parabolic differential equation.

I am very grateful to Heinz-Otto Kreiss who proposed this problem to me and who guided my work. He also gave me the opportunity to read the manuscript of the paper by P. D. Lax and B. Wendroff mentioned above, which the authors had kindly put at his disposal. I also want to thank Lars Hörmander for a clarifying discussion on Gårding's [7] device.

2. Proof of Theorem 1. First observe that because of (c) we only have to consider a small neighborhood of $\xi = 0$. Using (b) and (1.9) we see that $\gamma_1 = 1$. Since the eigenvalues depend continuously on the matrix, (b) also tells us that we only have to study the eigenvalues $\kappa_i(\xi)$ close to $\gamma_1 = 1$. Put $\eta = \kappa - 1$. Using (1.9) we get from (1.8):

$$(2.1) \quad \det \left\{ \eta \left(1 + \sum_{\mu=0}^r \alpha_{\mu\mu} \right) - \sum_{\mu=-1}^r \hat{Q}_{\mu}^{(1)} + O(\eta |\xi|^{2m}) + O(\eta^2) \right\} = 0.$$

(1.4) is a consistent approximation to (1.3). This easily establishes

$$(2.2) \quad \left(1 + \sum_{\mu=0}^r \mu \alpha_{\mu} \right) \lambda \sum_{|\nu|=2m} A_{\nu}(x) (i\xi_1)^{\nu_1} \cdots (i\xi_s)^{\nu_s} = \sum_{\mu=-1}^r \hat{Q}_{\mu}^{(1)} + o(|\xi|^{2m}).$$

Substitute (2.2) into (2.1). Using (1.1) and (1.2) we are able to prove (1.13) at once.

3. A Fundamental Inequality. The purpose of this section is to establish the following lemma.

LEMMA 1. *Suppose that the assumptions of Theorem 2 hold. Then, for each $x_0 \in R_s$, there exists a norm $\| \cdot \|_{H(x_0)}$ equivalent to $\| \cdot \|_r$ such that, for all $u \in S_r$:*

$$(3.1) \quad \| u \|_{H(x_0)}^2 - \| (\tilde{Q}^{(0)} + \tilde{Q}^{(1)}(x_0, D, h))u \|_{H(x_0)}^2 \geq \delta (L_{m, H(x_0)}(u))^2,$$

where $\delta > 0$ is a constant independent of x_0 , and

$$(3.2) \quad (L_{q, H(x_0)}(u))^2 = h^{2q} \sum_{j=1}^s \| D_{+j}^q u \|_{H(x_0)}^2,$$

where

$$C_{p+i} = \begin{pmatrix} \gamma_{p+i'} \cdot I & \epsilon I & & & \\ & \gamma_{p+i'} \cdot I & \epsilon I & & \\ & & \cdot & \cdot & \\ & & & \cdot & \\ & & & & \gamma_{p+i'} \cdot I \end{pmatrix}, \quad |\gamma_{p+i'}| < 1.$$

Taking into account that $\hat{Q}^{(1)} = O(|\xi|^{2m})$ it follows that

$$S(\hat{Q}^{(0)} + \hat{Q}^{(1)})S^{-1} = S\hat{Q}^{(0)}S^{-1} + O(|\xi|^{2m}).$$

Therefore by Gershgorin's estimate of the eigenvalues of a matrix we have

$$\begin{aligned} |\kappa_i - \gamma_{i'}| &\leq \text{const.} \cdot |\xi|^{2m} \quad \text{if} \quad |\gamma_{i'}| = 1, \\ |\kappa_i - \gamma_{i'}| &\leq \text{const.} \cdot |\xi|^{2m} + \epsilon \quad \text{if} \quad |\gamma_{i'}| < 1, \end{aligned} \quad \begin{aligned} i &= i' \cdot n - i'', \quad 0 \leq i'' < n. \end{aligned}$$

We can now write

$$sS(\hat{Q}^{(0)} + \hat{Q}^{(1)})S^{-1} - I = \begin{pmatrix} s\kappa_1 + sq_1 - 1 & sq_{12} & \dots & \dots & \dots & sq_{1m'} \\ sq_{21} & s\kappa_2 + sq_2 - 1 & \dots & \dots & \dots & sq_{2m'} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ sq_{i1} & \dots & s\kappa_i + sq_i - 1 + q_i' \epsilon & \dots & sq_{ii+n} + \epsilon q_i'' & \dots & sq_{im'} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ sq_{m'1} & \dots & \dots & \dots & \dots & \dots & s\kappa_{m'} + sq_{m'} - 1 + q_{m'} \epsilon \end{pmatrix},$$

where $m' = n(r + 1)$, $q_i' = q_i'' = 0$ for $i \leq pn$ and $|q_i'| \leq 1$, $q_i'' = 0$ or 1 for $i > pn$. At this

$$(3.8) \quad |q_i| + |q_{ik}| \leq \text{const.} \cdot |\xi|^{2m},$$

where because of the uniform bound for the coefficients of $\hat{Q}^{(1)}$, the constant can be chosen independent of x_0 . Denote by c_{kl} the elements of $(sS(\hat{Q}^{(0)} + \hat{Q}^{(1)})S^{-1} - I)^{-1}$ and by D_{kl} the matrix obtained from $sS(\hat{Q}^{(0)} + \hat{Q}^{(1)})S^{-1} - I$ by omitting the k th row and the l th column. Then by Cramer's rule:

$$(3.9) \quad c_{kl} = (-1)^{k+l} \det D_{lk} / \det(sS(\hat{Q}^{(0)} + \hat{Q}^{(1)})S^{-1} - I).$$

Observe that

$$\det(sS(\hat{Q}^{(0)} + \hat{Q}^{(1)})S^{-1} - I) = \prod_{i=1}^{m'} (s\kappa_i - 1).$$

From (1.13) and the assumptions of Theorem 2 follows:

$$(3.10) \quad \begin{aligned} |s\kappa_i - 1| &\geq 1 - |s\kappa_i| \geq \delta_1 |\xi|^{2m}, \quad i \leq pn, |s| < 1, \\ |s\kappa_i - 1| &\geq \text{const.}, \quad i > pn, |s| < 1, \end{aligned}$$

for sufficiently small $\epsilon > 0$. We also know that

$$(3.11) \quad |s\kappa_i - 1| \geq 1 - |s|.$$

Using elementary properties of a determinant we obtain from (3.9)

$$c_{kl} = (-1)^{k+l} \det E_{lk} / (s_{kl} - 1),$$

where we construct E_{lk} by dividing the l th row of $sS(\hat{Q}^{(0)} + \hat{Q}^{(1)})S^{-1} - I$ by $s_{kl} - 1$, respectively, and finally omitting the l th row and the k th column. Now use (3.8), (3.10) and (3.11). This easily shows that

$$|c_{kl}| \leq \text{const.}/(1 - |s|)$$

with the constant chosen independent of x_0 and ξ . Using Lemma 2 we obtain

$$|(\hat{Q}^{(0)} + \hat{Q}^{(1)}(x_0, \xi))^\nu| \leq C_1, \quad \nu = 1, 2, \dots,$$

or from (3.6), (3.7) and (1.13):

$$(3.12) \quad (\hat{Q}^{(0)} + \hat{Q}^{(1)}(x_0, \xi))^* \hat{H}(x_0, \xi) (\hat{Q}^{(0)} + \hat{Q}^{(1)}(x_0, \xi)) \\ \leq \hat{H}(x_0, \xi) (1 - \delta_1/2) |\xi|^{2m}$$

and

$$(3.13) \quad C_3^{-1}I \leq \hat{H}(x_0, \xi) \leq C_3I,$$

where $C_3 > 0$ is a constant independent of x_0 and $\hat{H}(x_0, \xi)$ is a positive definite hermitian matrix. Because of the fact that the matrix \mathcal{U} in Lemma 2 can be chosen as a piecewise continuous matrix-valued function of the elements of $\hat{Q}^{(0)} + \hat{Q}^{(1)}$ it follows from Lemma 2 that we are able to construct an operator $H(x_0)$ by

$$(3.14) \quad H(x_0)u(x) = (2\pi)^{-s/2} \int_{R_s} \hat{H}(x_0, \xi) e^{i\omega'x} \hat{u}(\omega) d\omega,$$

where

$$\hat{u}(\omega) = (2\pi)^{-s/2} \int_{R_s} e^{-i\omega'x} u(x) dx.$$

We now define the new norm by

$$(3.15) \quad \|u\|_{H(x_0)}^2 = (u, H(x_0)u)_r.$$

Using Parseval's relation and (3.13) we obtain (3.3). Observe that

$$(3.16) \quad (L_{q,H(x_0)}(u))^2 = \int_{R_s} \sum_j (2 \sin \xi_j/2)^{2q} \hat{u}^*(\omega) \hat{H}(x_0, \xi) \hat{u}(\omega) d\omega,$$

and that

$$(3.17) \quad |\xi_j| \geq |2 \sin \xi_j/2|.$$

(3.1) is now established using Parseval's relation, (3.12) and (3.17).

4. Some Lemmas. Here we collect some lemmas, which we need in Section 5.

LEMMA 3. *There exists a function $\phi(x) \in C^\infty(R_s)$ such that,⁴ the support of $\phi(x)$ is contained in the cube defined by $\max_\nu |x_\nu| < 1$, $\phi(x) \geq 0$ and*

$$(4.1) \quad 1 = \sum_g \phi^2(x - g),$$

⁴ By $C^\infty(R_s)$ we denote the set of all functions defined in R_s , whose partial derivatives of all orders exist.

where g runs through all points $g = (g_1, \dots, g_s)$ with integral coordinates.

Proof. Hörmander [8] showed that there exists a non-negative function $\theta(x) \in C^\infty(R_s)$ with support contained in the cube $\max_v |x_v| < 1$ such that $\theta(x) \geq 0$ and

$$1 = \sum_g \theta(x - g).$$

Then

$$\phi(x) = \theta(x) / \sqrt{\sum_g \theta^2(x - g)}$$

is easily seen to have the required properties.

For a given $\epsilon > 0$ and any $u(x) \in S_r$, now define:

$$(4.2) \quad u_g(x) = \phi((x - \epsilon g)/\epsilon)u(x).$$

Observe that at most 2^s vector-valued functions u_g can be different from 0 at any point $x \in R_s$.

LEMMA 4. *Let*

$$(4.3) \quad (u, Hv) = \sum_g (u_g, H(x_g)v_g)_r, \quad u, v \in S_r, x_g = \epsilon g,$$

where u_g and v_g are defined by (4.2) and $H(x_g)v_g$ is defined by (3.14). Then (u, Hv) is an inner product defining an equivalent norm which satisfies

$$(4.4) \quad C_3^{-1} \|u\|_r^2 \leq (u, Hu) \leq C_3 \|u\|_r^2$$

where C_3 is the constant in (3.3).

Proof. (u, Hv) is easily seen to be an inner product. (4.4) follows from (3.3) and (4.1).

LEMMA 5. *Let p and q , $p \geq q$, be positive integers. Then*

$$L_{p,H(x_g)}(u) \leq 2^{p-q} L_{q,H(x_g)}(u).$$

Proof. The inequality follows from:

$$\begin{aligned} ((hD_{+j}(h))^p u, H(x_g)(hD_{+j}(h))^p u)_r &= \int_{R_s} \hat{u}^* \hat{H}(x_g) \hat{u} (2 \sin \xi_j/2)^{2p} d\omega \\ &\leq \int_{R_s} 4^{p-q} \hat{u}^* \hat{H}(x_g) \hat{u} (2 \sin \xi_j/2)^{2q} d\omega \\ &= 4^{p-q} ((hD_{+j}(h))^q u, H(x_g)(hD_{+j}(h))^q u)_r. \end{aligned}$$

LEMMA 6. *For any $\epsilon_1 > 0$ there exists a constant $C(\epsilon_1)$ such that if $p > q$:*

$$h^{2p-2q} (L_{q,H(x_g)}(u))^2 \leq \epsilon_1 (L_{p,H(x_g)}(u))^2 + C(\epsilon_1) h^{2p} (L_{0,H(x_g)}(u))^2.$$

Proof. The lemma follows from the following elementary inequality:

$$h^{2p-2q} (2 \sin \xi_j/2)^{2q} \leq \epsilon_1 (2 \sin \xi_j/2)^{2p} + C(\epsilon_1) h^{2p}.$$

Observing that the difference operators commute with $H(x_g)$ we establish the following lemma at once.

LEMMA 7.

$$\begin{aligned} (u, H(x_\sigma)A(hD_{0j}(h)v))_r &= -(hD_{0j}(h)u, H(x_\sigma)Av)_r \\ &\quad - h/2(E_j^{-1}u, H(x_\sigma)(E_j^{-\theta_1}\partial A/\partial x_j)v)_r \\ &\quad - h/2(E_j u, H(x_\sigma)(E_j^{\theta_2}\partial A/\partial x_j)v)_r, \\ (u, H(x_\sigma)AhD_{+j}(h)v)_r &= -(hD_{-j}(h)u, H(x_\sigma)Av)_r \\ &\quad - (E_j^{-1}u, H(x_\sigma)(E_j^{-\theta_3}\partial A/\partial x_j)v)_r, \end{aligned}$$

where $0 < \theta_1, \theta_2, \theta_3 < 1$, $A(x)$ is a sufficiently smooth matrix and $E_j u(x) = u(x + he_j)$.

LEMMA 8. Let $(hD)^p$ denote the product of p operators hD , where each hD stands for one of the operators hD_{0j} , hD_{+j} or hD_{-j} . Then

$$L_{0,H(x_\sigma)}((hD)^p u) \leq L_{p,H(x_\sigma)}(u).$$

Proof.

$$\begin{aligned} (L_{0,H(x_\sigma)}((hD)^p u))^2 &\leq \int_{R_\sigma} \hat{u}^* \hat{H}(x_\sigma) \hat{u} (\max(2 \sin \xi_j/2))^{2p} d\omega \\ &\leq \sum_j \int_{R_\sigma} \hat{u}^* \hat{H}(x_\sigma) \hat{u} (2 \sin \xi_j/2)^{2p} d\omega = (L_{p,H(x_\sigma)}(u))^2. \end{aligned}$$

LEMMA 9. Let $(hD)^p$ be defined as in Lemma 8. Denote by $(\overline{hD})^p$ the operator constructed by exchanging hD_{+j} for hD_{-j} and hD_{-j} for hD_{+j} in $(hD)^p$. Let $A(x)$ be a sufficiently smooth matrix such that the derivatives of order $\leq p_1 \leq m$ of the elements of $A(x)$ are uniformly bounded. For any $\epsilon_2 > 0$ there exists a constant $C(\epsilon_2)$ independent of h, g, u and v such that

$$\begin{aligned} h^{m-p_1} | (u, H(x_\sigma)A(hD)^{p_1}(hD)^m v)_r - (-1)^{p_1} (\overline{hD})^{p_1} u, H(x_\sigma)A(hD)^m v)_r | \\ \leq \epsilon_2 ((L_{m,H(x_\sigma)}(u))^2 + (L_{m,H(x_\sigma)}(v))^2) + C(\epsilon_2) ((L_{0,H(x_\sigma)}(u))^2 + (L_{0,H(x_\sigma)}(v))^2) h^{2m}. \end{aligned}$$

Proof. Using Lemma 7, Lemma 8 and Schwarz's inequality several times it is possible to show that

$$\begin{aligned} | (u, H(x_\sigma)A(hD)^{p_1}(hD)^m v)_r - (-1)^{p_1} (\overline{hD})^{p_1} u, H(x_\sigma)A(hD)^m v)_r | \\ \leq \text{const.} \sum_{\nu_2 \leq m; \nu_1 < p_1} h^{p_1+m-(\nu_1+\nu_2)} L_{\nu_1,H(x_\sigma)}(u) L_{\nu_2,H(x_\sigma)}(v). \end{aligned}$$

We now use the elementary inequality

$$ab \leq a^2/2c + b^2c/2$$

and Lemma 5 and Lemma 6 to finish the proof.

5. Proof of Theorem 2. The proof is carried out in three steps. At this we use arguments analogous to those of Hörmander [8].

First consider:

$$\begin{aligned} (5.1) \quad (L_{0,H(x_\sigma)}(u_\sigma))^2 - (L_{0,H(x_\sigma)}(\tilde{Q}(x_\sigma, D, h)u_\sigma))^2 \\ = (L_{0,H(x_\sigma)}(u_\sigma))^2 - (L_{0,H(x_\sigma)}((\tilde{Q}^{(0)} + \tilde{Q}^{(1)}(x_\sigma, D, h))u_\sigma))^2 - \text{I} - \text{II} - \text{III}. \end{aligned}$$

Here

$$\begin{aligned} \text{I} &= 2 \operatorname{Re}(\tilde{Q}^{(0)}u_g, H(x_g)\tilde{Q}^{(2)}(x_g, D, h)u_g)_r, \\ \text{II} &= 2 \operatorname{Re}(\tilde{Q}^{(1)}(x_g, D, h)u_g, H(x_g)\tilde{Q}^{(2)}(x_g, D, h)u_g)_r, \\ \text{III} &= (L_{0,H(x_g)}(\tilde{Q}^{(2)}(x_g, D, h)u_g))^2. \end{aligned}$$

Now observe that from the uniform boundedness of $(I - Q_{-1}(x_g, D, h))^{-1}$ follows,
 $\| (I_{\pm}^{\pm} - Q_{-1}^{(1)}(x_g, D, h))v \|_0 + \| Q_{-1}(x_g, D, h)v - Q_{-1}^{(1)}(x_g, D, h)v \|_0$
 $\geq \| (I - Q_{-1}(x_g, D, h))v \|_0 \geq \text{const.} \| v \|_0.$

Using Lemma 5 we easily show that

$$\| Q_{-1}(x_g, D, h)v - Q_{-1}^{(1)}(x_g, D, h)v \|_0 \leq h \text{const.} \| v \|_0.$$

This shows that $(I - Q_{-1}^{(1)}(x_g, D, h))^{-1}$ is a uniformly bounded operator if h is sufficiently small. Also observe that

$$\begin{aligned} &(I - Q_{-1})^{-1}(\alpha_{\mu}I + Q_{\mu}) - (I - Q_{-1}^{(1)})^{-1}(\alpha_{\mu}I + Q_{\mu}^{(1)}) \\ &= (I - Q_{-1})^{-1}(Q_{-1} - Q_{-1}^{(1)})(I - Q_{-1}^{(1)})^{-1}(\alpha_{\mu}I + Q_{\mu}^{(1)}) + (I - Q_{-1})^{-1}(Q_{\mu} - Q_{\mu}^{(1)}). \end{aligned}$$

Here

$$Q_{\mu} - Q_{\mu}^{(1)} = \sum_{\sigma>0; \tau+\sigma \geq 2m} B_{\mu}^{\sigma\tau} h^{\sigma} (hD)^{\tau}.$$

Using Schwarz's inequality, Lemmas 5, 6, 8 and 9 we easily show that,

$$(5.2) \quad |\text{I}| + |\text{II}| + |\text{III}| \leq \beta' k(L_{0,H(x_g)}(u_g))^2 + (\delta/2)(L_{m,H(x_g)}(u_g))^2,$$

where β' is a constant independent of g . Combine (5.1) and (5.2) and use Lemma 1:

$$(5.3) \quad \begin{aligned} &(L_{0,H(x_g)}(u_g))^2 - (L_{0,H(x_g)}(\tilde{Q}(x_g, D, h)u_g))^2 \\ &\geq -|\beta'| k(L_{0,H(x_g)}(u_g))^2 + (\delta/2)(L_{m,H(x_g)}(u_g))^2. \end{aligned}$$

Now consider:

$$\begin{aligned} &(L_{0,H(x_g)}(u_g))^2 - (L_{0,H(x_g)}(\tilde{Q}(x, D, h)u_g))^2 \\ &= (L_{0,H(x_g)}(u_g))^2 - (L_{0,H(x_g)}(\tilde{Q}(x_g, D, h)u_g))^2 + \text{IV} + \text{V} + \text{VI}. \end{aligned}$$

Here

$$\begin{aligned} \text{IV} &= 2 \operatorname{Re}(\tilde{Q}^{(0)}u_g, H(x_g)((\tilde{Q}(x_g, D, h) - \tilde{Q}(x, D, h))u_g))_r, \\ \text{V} &= 2 \operatorname{Re}((\tilde{Q}^{(1)}(x_g, D, h) + \tilde{Q}^{(2)}(x_g, D, h))u_g, H(x_g)(\tilde{Q}(x_g, D, h) - \tilde{Q}(x, D, h))u_g)_r, \\ \text{VI} &= -(L_{0,H(x_g)}((\tilde{Q}(x, D, h) - \tilde{Q}(x_g, D, h))u_g))^2. \end{aligned}$$

Now choose ϵ in (4.2) sufficiently small. We can then estimate IV using Lemma 9, Schwarz's inequality, Lemmas 5, 6 and 8 obtaining:

$$|\text{IV}| \leq \beta'' k(L_{0,H(x_g)}(u_g))^2 + (\delta/12)(L_{m,H(x_g)}(u_g))^2.$$

It is easy to establish the same type of estimates for V and VI. Thus, using (5.3)

$$(5.4) \quad \begin{aligned} &(L_{0,H(x_g)}(u_g))^2 - (L_{0,H(x_g)}(\tilde{Q}(x, D, h)u_g))^2 \\ &\geq (\delta/4)(L_{m,H(x_g)}(u_g))^2 + \beta''' k(L_{0,H(x_g)}(u_g))^2. \end{aligned}$$

Finally consider:

$$\begin{aligned} & (u, Hu) - (\tilde{Q}(x, D, h)u, H\tilde{Q}(x, D, h)u) \\ &= \sum_{\sigma} (u_{\sigma}, H(x_{\sigma})u_{\sigma})_r - ((\tilde{Q}(x, D, h)u)_{\sigma}, H(x_{\sigma})(\tilde{Q}(x, D, h)u)_{\sigma})_r \\ &= \sum_{\sigma} (u_{\sigma}, H(x_{\sigma})u_{\sigma})_r - (\tilde{Q}(x, D, h)u_{\sigma}, H(x_{\sigma})\tilde{Q}(x, D, h)u_{\sigma})_r - \text{VII} - \text{VIII}. \end{aligned}$$

Here

$$\begin{aligned} \text{VII} &= 2 \operatorname{Re} \sum_{\sigma} (\tilde{Q}(x, D, h)u_{\sigma}, H(x_{\sigma})(\tilde{Q}(x, D, h)u)_{\sigma} - \tilde{Q}(x, D, h)u_{\sigma})_r, \\ \text{VIII} &= \sum_{\sigma} (L_{0,H(x_{\sigma})}((\tilde{Q}(x, D, h)u)_{\sigma} - \tilde{Q}(x, D, h)u_{\sigma}))^2. \end{aligned}$$

Because of the fact that all derivatives of $\phi(x)$ are uniformly bounded we easily show that,

$$(\tilde{Q}(x, D, h)u)_{\sigma} - \tilde{Q}(x, D, h)u_{\sigma} = h \sum_{\tau, \nu} C_{\tau, \nu, \sigma}(x) h^{\tau} (hD)^{\nu} u,$$

where the elements of $C_{\tau, \nu, \sigma}(x)$ have sufficiently many uniformly bounded derivatives. Observe that there exists a constant $N(h)$, decreasing with decreasing h , so that no more than $N(h)$ of the matrices $C_{\tau, \nu, \sigma}(x)$ are different from 0 at any point $x \in R_{\sigma}$. Using this, (5.4), and the same type of arguments as above we are now able to establish that

$$(u, Hu) - (\tilde{Q}(x, D, h)u, H\tilde{Q}(x, D, h)u) \geq \beta^{IV} k(u, Hu).$$

This proves Theorem 2.

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