

# A New Integral Representation of the Bessel Coefficients

By P. Razelos

The modified Bessel coefficients  $I_n$  are defined by the series [1]

$$(1) \quad I_n(t) = \sum_{q=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2q+n}}{q!(q+n)!},$$

with the integral representation [1]

$$(2) \quad I_n(t) = \frac{1}{\pi} \int_0^{\pi} e^{t \cos x} \cos nx \, dx.$$

An integral representation of the coefficients  $I_n(t)$  is presented here where the path of integration is extended to infinity.

**THEOREM:** *The integral*

$$A_n(t) = \frac{2}{\pi} \int_0^{\infty} e^{t \cos x} \cos nx \frac{\sin \epsilon x}{x} \, dx$$

is the modified coefficient  $I_n(t)$  for any  $0 < \epsilon < 1$ . There are many ways by which the theorem can be proved, but we give here the following proof which consists of a straightforward evaluation of the integral  $A_n$ . The function  $e^{t \cos x}$  is expanded in a power series (which is absolutely convergent for all  $t$ ) and we then integrate term by term.

$$(3) \quad e^{t \cos x} = \sum_{r=0}^{\infty} \frac{(t \cos x)^r}{r!}.$$

Let us define

$$(4) \quad Q_n^r = \frac{2}{\pi} \int_0^{\infty} \cos^r x \cos nx \frac{\sin \epsilon x}{x} \, dx.$$

Then

$$(5) \quad A_n(t) = \sum_{r=0}^{\infty} \frac{Q_n^r}{r!} t^r.$$

Introducing the expansion

$$(6) \quad \cos^r x = \frac{1}{2^{r-1}} \sum_{k=0}^{(r/2)-\delta} \binom{r}{k} \cos(r-2k)x$$

(where  $\delta = 1$  or  $\frac{1}{2}$  for  $r$  even or odd, respectively) into (4), we obtain

$$(7) \quad Q_n^r = \frac{1}{2^r} \sum_{k=0}^{(r/2)-\delta} \binom{r}{k} [g(r-2k+n) + g(r-2k-n)],$$

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Received March 31, 1964. Revised July 6, 1964.

where

$$(8) \quad g(y) = \frac{2}{\pi} \int_0^\infty \cos yx \frac{\sin \epsilon x}{x} dx = 0, 1, \frac{1}{2}$$

for  $|y/\epsilon|$  greater than, less than, or equal to one, respectively [2]. Therefore, the only term which is nonzero in (7) is the term  $g(0)$ , if it exists.

Then

$$(9) \quad Q_n^r = \frac{1}{2^r} \frac{r!}{\left(\frac{r-n}{2}\right)! \left(\frac{r+n}{2}\right)!}, \quad r \geq n \text{ and } r, n \text{ both even or odd,}$$

$$(10) \quad Q_n^r = 0 \text{ for } r < n \text{ or } r, n \text{ one even, one odd.}$$

We can now write

$$(11) \quad \begin{aligned} r - n &= 2q, \\ r + n &= 2(q + n). \end{aligned}$$

Thus,

$$(12) \quad Q_n^r = \frac{r!}{2^r q!(q+n)!}.$$

Substituting (12) into (5), we obtain

$$(13) \quad A_n(t) = \sum_{q=0}^\infty \frac{\left(\frac{t}{2}\right)^{2q+n}}{q!(q+n)!} = I_n. \quad \text{Q.E.D.}$$

A similar expression can be readily obtained for the coefficients  $J_n(t)$ . The value of  $\epsilon = \frac{1}{2}$  gives the following interesting result. Let us define

$$(14) \quad B_n(t) = \frac{2}{\pi} \int_{n-(1/2)}^{n+(1/2)} \int_0^\infty e^{t \cos x} \cos \nu x \frac{\sin(x/2)}{x} d\nu dx.$$

It can be easily shown that  $B_n(t) = I_n(t)$ . Consider now the integral

$$\begin{aligned} B(t) &\doteq \frac{2}{\pi} \int_{-\infty}^\infty I_\nu\left(\frac{1}{2}, t\right) d\nu \\ &= \frac{2}{\pi} \int_0^\infty d\nu \int_0^\infty 2e^{t \cos x} \cos(\nu x) \frac{\sin(x/2)}{x} dx \\ &= \lim_{\nu \rightarrow 0} \frac{2}{\pi} \int_0^\infty d\nu \int_0^\infty 2e^{t \cos x} \cos(\nu x) \cos(\nu y) \frac{\sin(x/2)}{x} dx \\ &= 2e^{t \cos y} \frac{\sin y/2}{y} \Big|_{y=0} = e^t \end{aligned}$$

by Fourier's theorem. Clearly  $B(t) = \sum_{-\infty}^\infty B_n(t)$ .

**Acknowledgment.** The author wishes to thank Professor H. G. Elrod, Jr. for suggesting the problem.

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1. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, New York, 1944. MR 6, 64.

2. R. S. BURINGTON, *Handbook of Mathematical Tables and Formulas*, 3rd ed., Handbook Publishers Inc., Sandusky, Ohio, 1949. MR 22 #2514.

## Footnote to the Evaluation of Certain Complex Elliptic Integrals

By C. D. Sutherland

The formulas for evaluating the elliptic integral of the third kind with a complex parameter as given by Byrd and Friedman [1] have been corrected and simplified by Lang and Stevens [2]. There is, however, a further correction necessary in these latter results.

The integral to be evaluated is

$$I = (a_1 + ib_1) \int_0^\phi \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\Delta},$$

where  $\alpha^2$  is complex and  $\Delta = \sqrt{(1 - k^2 \sin^2 \theta)}$ . In the formulas for evaluating  $I$  there appears the quantity

$$\tau_2 = \int_0^{p_2} \frac{m_2 dx}{1 + h_2 x^2} = \frac{m_2}{\sqrt{h_2}} \tan^{-1}(p_2 \sqrt{h_2}),$$

where

$$p_2 = \frac{\sin \phi \cos \phi}{(1 + m_2 \sin^2 \phi)\Delta}.$$

We will consider the case where  $m_2 \leq -1$ . If this occurs we see that as  $\phi$  goes to  $\pi/2$ , either  $p_2 \rightarrow \infty$  ( $m_2 = -1$ ) and  $[\tan^{-1}(p_2 \sqrt{h_2})] \rightarrow \pi/2$ , or  $p_2 \rightarrow 0$  through negative values ( $m_2 < -1$ ) and  $[\tan^{-1}(p_2 \sqrt{h_2})] \rightarrow \pi$  (and not to zero). To avoid overlooking this possibility the proper representation for  $\tau_2$  is

$$\tau_2 = \frac{-1}{\sqrt{h_2}} \cos^{-1} \left( \frac{\Delta \cos \phi}{\sqrt{(h_2 \sin^2 \phi + \Delta^2 \cos^2 \phi)}} \right) \quad \text{for } m_2 = -1,$$

$$\tau_2 = \frac{m_2}{\sqrt{h_2}} \cos^{-1} \left( \frac{\Delta(1 + m_2 \sin^2 \phi)}{\sqrt{(h_2 \sin^2 \phi \cos^2 \phi + \Delta^2(1 + m_2 \sin^2 \phi)^2)}} \right) \quad \text{for } m_2 \neq -1.$$

It is to be noted, in particular, that the formulas for the real and imaginary parts of the complete integral should contain a term involving  $\tau_2$  whenever  $m_2 \leq -1$ .

Received March 16, 1964. Work performed under the auspices of the U. S. Atomic Energy Commission.