A New Integral Representation of the Bessel Coefficients

By P. Razelos

The modified Bessel coefficients $I_n$ are defined by the series [1]

\[ I_n(t) = \sum_{q=0}^{\infty} \frac{(t/2)^{2q+n}}{q!(q+n)!}, \]

with the integral representation [1]

\[ I_n(t) = \frac{1}{\pi} \int_0^{\infty} e^{-tx} \cos nx \, dx. \]

An integral representation of the coefficients $I_n(t)$ is presented here where the path of integration is extended to infinity.

**Theorem:** The integral

\[ A_n(t) = \frac{1}{\pi} \int_0^{\infty} e^{-t \cos x} \cos nx \, dx \]

is the modified coefficient $I_n(t)$ for any $0 < t < 1$. There are many ways by which the theorem can be proved, but we give here the following proof which consists of a straightforward evaluation of the integral $A_n$. The function $e^{-t \cos x}$ is expanded in a power series (which is absolutely convergent for all $t$) and we then integrate term by term.

\[ e^{-t \cos x} = \sum_{r=0}^{\infty} \frac{(t \cos x)^r}{r!}. \]

Let us define

\[ Q_n^r = \frac{2}{\pi} \int_0^{\infty} \cos^r x \cos nx \frac{\sin \varepsilon x}{x} \, dx. \]

Then

\[ A_n(t) = \sum_{r=0}^{\infty} \frac{Q_n^r}{r!} t^r. \]

Introducing the expansion

\[ \cos^r x = \frac{1}{2^{r-1}} \sum_{k=0}^{(r/2)-\delta} \binom{r}{k} \cos(r - 2k)x \]

(where $\delta = 1$ or $\frac{1}{2}$ for $r$ even or odd, respectively) into (4), we obtain

\[ Q_n^r = \frac{1}{2^r} \sum_{k=0}^{(r/2)-\delta} \binom{r}{k} \{g(r - 2k + n) + g(r - 2k - n)\}, \]

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where

\[(8) \quad g(y) = \frac{2}{\pi} \int_0^\infty \cos yx \frac{\sin \varepsilon x}{x} dx = 0, 1, \frac{1}{2}\]

for \(|y/\varepsilon|\) greater than, less than, or equal to one, respectively [2]. Therefore, the only term which is nonzero in (7) is the term \(g(0)\), if it exists.

Then

\[(9) \quad Q_{n^r} = \frac{1}{2^r} \frac{r!}{\left(\frac{r-n}{2}\right)!\left(\frac{r+n}{2}\right)!}, \quad r \geq n \text{ and } r, n \text{ both even or odd},\]

\[(10) \quad Q_{n^r} = 0 \quad \text{for } r < n \quad \text{or} \quad r, n \text{ one even, one odd}.

We can now write

\[(11) \quad r - n = 2q, \quad r + n = 2(q + n).\]

Thus,

\[(12) \quad Q_{n^r} = \frac{r!}{2^r q! (q + n)!}.\]

Substituting (12) into (5), we obtain

\[(13) \quad A_n(t) = \sum_{q=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2q+n}}{q! (q + n)!} = I_n. \quad \text{Q.E.D.}\]

A similar expression can be readily obtained for the coefficients \(J_n(t)\). The value of \(\varepsilon = \frac{1}{2}\) gives the following interesting result. Let us define

\[(14) \quad B_n(t) = \frac{2}{\pi} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \int_0^\infty e^{t \cos x} \cos \nu x \frac{\sin(x/2)}{x} dv dx.\]

It can be easily shown that \(B_n(t) = I_n(t)\). Consider now the integral

\[B(t) = \frac{2}{\pi} \int_{-\infty}^{\infty} I_* \left(\frac{1}{2}, t\right) dv\]

\[= \frac{2}{\pi} \int_0^\infty dv \int_0^\infty 2e^{t \cos x} \cos(\nu x) \sin(x/2) x dx\]

\[= \lim_{\nu \to 0} \frac{2}{\pi} \int_0^\infty dv \int_0^\infty 2e^{t \cos x} \cos(\nu x) \cos(\nu y) \sin(x/2) x dx\]

\[= 2e^{t \cos \frac{y}{2}} \bigg|_{y=0}^{y=\infty} = e^t\]

by Fourier's theorem. Clearly \(B(t) = \sum_{n=0}^{\infty} B_n(t)\).
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Footnote to the Evaluation of Certain Complex Elliptic Integrals

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The formulas for evaluating the elliptic integral of the third kind with a complex parameter as given by Byrd and Friedman [1] have been corrected and simplified by Lang and Stevens [2]. There is, however, a further correction necessary in these latter results.

The integral to be evaluated is

$$ I = (a_1 + ib_1) \int_0^\phi \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\Delta}, $$

where $\alpha^2$ is complex and $\Delta = \sqrt{(1 - k^2 \sin^2 \theta)}$. In the formulas for evaluating $I$ there appears the quantity

$$ \tau_2 = \int_0^{p_2} \frac{m_2 \, dx}{1 + h_2 \, x^2} = \frac{m_2}{\sqrt{h_2}} \tan^{-1}(p_2\sqrt{h_2}), $$

where

$$ p_2 = \frac{\sin \phi \cos \phi}{(1 + m_2 \sin^2 \phi)\Delta}. $$

We will consider the case where $m_2 \leq -1$. If this occurs we see that as $\phi$ goes to $\pi/2$, either $p_2 \to \infty$ ($m_2 = -1$) and $[\tan^{-1}(p_2\sqrt{h_2})] \to \pi/2$, or $p_2 \to 0$ through negative values ($m_2 < -1$) and $[\tan^{-1}(p_2\sqrt{h_2})] \to \pi$ (and not to zero). To avoid overlooking this possibility the proper representation for $\tau_2$ is

$$ \tau_2 = \frac{-1}{\sqrt{h_2}} \cos^{-1} \left( \frac{\Delta \cos \phi}{\sqrt{(h_2 \sin^2 \phi + \Delta^2 \cos^2 \phi)}} \right) \quad \text{for} \quad m_2 = -1, $$

$$ \tau_2 = \frac{m_2}{\sqrt{h_2}} \cos^{-1} \left( \frac{\Delta(1 + m_2 \sin^2 \phi)}{\sqrt{(h_2 \sin^2 \phi \cos^2 \phi + \Delta^2(1 + m_2 \sin^2 \phi)^2)}} \right) \quad \text{for} \quad m_2 \neq -1. $$

It is to be noted, in particular, that the formulas for the real and imaginary parts of the complete integral should contain a term involving $\tau_2$ whenever $m_2 \leq -1$.

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