A New Integral Representation of the Bessel Coefficients

By P. Razelos

The modified Bessel coefficients I_n are defined by the series [1]

$$(1)\qquad I_n(t) = \sum_{q=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2q+n}}{q!(q+n)!},$$

with the integral representation [1]

$$(2)\qquad I_n(t) = \frac{1}{\pi}\int_0^{\pi} e^{t\cos x}\cos nx\,dx.$$

An integral representation of the coefficients $I_n(t)$ is presented here where the path of integration is extended to infinity.

Theorem: *The integral*

$$A_n(t) = \frac{2}{\pi}\int_0^{\infty} e^{t\cos x}\cos nx\,\frac{\sin \epsilon x}{x}\,dx$$

is the modified coefficient $I_n(t)$ *for any* $0 < \epsilon < 1$. There are many ways by which the theorem can be proved, but we give here the following proof which consists of a straightforward evaluation of the integral A_n. The function $e^{t\cos x}$ is expanded in a power series (which is absolutely convergent for all t) and we then integrate term by term.

$$(3)\qquad e^{t\cos x} = \sum_{r=0}^{\infty} \frac{(t\cos x)^r}{r!}.$$

Let us define

$$(4)\qquad Q_n^r = \frac{2}{\pi}\int_0^{\infty} \cos^r x\,\cos nx\,\frac{\sin \epsilon x}{x}\,dx.$$

Then

$$(5)\qquad A_n(t) = \sum_{r=0}^{\infty} \frac{Q_n^r}{r!}t^r.$$

Introducing the expansion

$$(6)\qquad \cos^r x = \frac{1}{2^{r-1}}\sum_{k=0}^{(r/2)-\delta}\binom{r}{k}\cos(r-2k)x$$

(where $\delta = 1$ or $\frac{1}{2}$ for r even or odd, respectively) into (4), we obtain

$$(7)\qquad Q_n^r = \frac{1}{2^r}\sum_{k=0}^{(r/2)-\delta}\binom{r}{k}[g(r-2k+n)+g(r-2k-n)],$$

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where

$$g(y) = \frac{2}{\pi}\int_0^\infty \cos yx \,\frac{\sin \epsilon x}{x}\, dx = 0, 1, \frac{1}{2} \tag{8}$$

for $|\, y/\epsilon \,|$ greater than, less than, or equal to one, respectively [2]. Therefore, the only term which is nonzero in (7) is the term $g(0)$, if it exists.

Then

$$Q_n^{\,r} = \frac{1}{2^r}\frac{r!}{\left(\dfrac{r-n}{2}\right)!\left(\dfrac{r+n}{2}\right)!}, \qquad r \geqq n \text{ and } r,\, n \text{ both even or odd,} \tag{9}$$

$$Q_n^{\,r} = 0 \quad \text{for } r < n \quad \text{or} \quad r,\, n \text{ one even, one odd.} \tag{10}$$

We can now write

$$\begin{aligned} r - n &= 2q, \\ r + n &= 2(q + n). \end{aligned} \tag{11}$$

Thus,

$$Q_n^{\,r} = \frac{r!}{2^r q!(q+n)!}. \tag{12}$$

Substituting (12) into (5), we obtain

$$A_n(t) = \sum_{q=0}^{\infty} \frac{\left(\dfrac{t}{2}\right)^{2q+n}}{q!(q+n)!} = I_n. \qquad \text{Q.E.D.} \tag{13}$$

A similar expression can be readily obtained for the coefficients $J_n(t)$. The value of $\epsilon = \frac{1}{2}$ gives the following interesting result. Let us define

$$B_n(t) = \frac{2}{\pi}\int_{n-(1/2)}^{n+(1/2)} \int_0^\infty e^{t\cos x} \cos \nu x \,\frac{\sin(x/2)}{x}\, d\nu dx. \tag{14}$$

It can be easily shown that $B_n(t) = I_n(t)$. Consider now the integral

$$\begin{aligned} B(t) &\doteq \frac{2}{\pi}\int_{-\infty}^{\infty} I_\nu\left(\frac{1}{2}, t\right) d\nu \\ &= \frac{2}{\pi}\int_0^\infty d\nu \int_0^\infty 2e^{t\cos x}\cos(\nu x)\,\frac{\sin(x/2)}{x}\, dx \\ &= \lim_{y\to 0}\frac{2}{\pi}\int_0^\infty d\nu \int_0^\infty 2e^{t\cos x}\cos(\nu x)\cos(\nu y)\,\frac{\sin(x/2)}{x}\, dx \\ &= 2e^{t\cos y}\,\frac{\sin y/2}{y}\bigg|_{y=0} = e^t \end{aligned}$$

by Fourier's theorem. Clearly $B(t) = \sum_{-\infty}^{\infty} B_n(t)$.

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1. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, New York, 1944. MR **6**, 64.
2. R. S. BURINGTON, *Handbook of Mathematical Tables and Formulas*, 3rd ed., Handbook Publishers Inc., Sandusky, Ohio, 1949. MR **22** #2514.

Footnote to the Evaluation of Certain Complex Elliptic Integrals

By C. D. Sutherland

The formulas for evaluating the elliptic integral of the third kind wi h a complex parameter as given by Byrd and Friedman [1] have been corrected and simplified by Lang and Stevens [2]. There is, however, a further correction necessary in these latter results.

The integral to be evaluated is

$$I = (a_1 + ib_1) \int_0^{\phi} \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\Delta},$$

where α^2 is complex and $\Delta = \sqrt{(1 - k^2 \sin^2 \theta)}$. In the formulas for evaluating I there appears the quantity

$$\tau_2 = \int_0^{p_2} \frac{m_2\, dx}{1 + h_2 x^2} = \frac{m_2}{\sqrt{h_2}} \tan^{-1}(p_2\sqrt{h_2}),$$

where

$$p_2 = \frac{\sin \phi \cos \phi}{(1 + m_2 \sin^2 \phi)\Delta}.$$

We will consider the case where $m_2 \leqq -1$. If this occurs we see that as ϕ goes to $\pi/2$, either $p_2 \to \infty$ $(m_2 = -1)$ and $[\tan^{-1}(p_2\sqrt{h_2})] \to \pi/2$, or $p_2 \to 0$ through negative values $(m_2 < -1)$ and $[\tan^{-1}(p_2\sqrt{h_2})] \to \pi$ (and not to zero). To avoid overlooking this possibility the proper representation for τ_2 is

$$\tau_2 = \frac{-1}{\sqrt{h_2}} \cos^{-1}\left(\frac{\Delta \cos \phi}{\sqrt{(h_2 \sin^2 \phi + \Delta^2 \cos^2 \phi)}}\right) \quad \text{for} \quad m_2 = -1,$$

$$\tau_2 = \frac{m_2}{\sqrt{h_2}} \cos^{-1}\left(\frac{\Delta(1 + m_2 \sin^2 \phi)}{\sqrt{(h_2 \sin^2 \phi \cos^2 \phi + \Delta^2(1 + m_2 \sin^2 \phi)^2)}}\right) \quad \text{for} \quad m_2 \neq -1.$$

It is to be noted, in particular, that the formulas for the real and imaginary parts of the complete integral should contain a term involving τ_2 whenever $m_2 \leqq -1$.

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