Bounds for the Spectral Radius of a Matrix

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Let \( A = [a_{ij}] \) be an \( n \times n \) matrix with complex entries. We define \( \rho(A) \) to be the spectral radius of \( A \) and \( |A| \) to be the matrix \( |a_{ij}| \).

A. Brauer [1], W. Ledermann [2] and A. Ostrowski [4] have developed bounds for \( \rho(| A |) \). Their results, coupled with the result of Perron and Frobenius [6] that \( \rho(A) \leq \rho(| A |) \) give upper bounds for \( \rho(A) \) which are not less than \( \rho(| A |) \). These bounds are restricted to matrices with nonzero entries and do not take into account the effect of the phases of the entries of \( A \) on \( \rho(A) \). In Section I of this paper we obtain a sequence of bounds for \( \rho(A) \) in terms of \( \rho(| A^r |) \) \((r = 1, 2, \ldots)\) which are less than or equal to \( \rho(| A |) \) and converge to \( \rho(A) \). In this manner we are partially accounting for the effect on \( \rho(A) \) of the phases of the \( a_{ij} \). In Section II we derive bounds for \( \rho(A) \) in terms of the Frobenius norm of \( A \). These bounds always lie in the field of values of \( A \), are computationally well suited to complex matrices and can be used in conjunction with the techniques of Section I.

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I. Bounds for \( \rho(A) \). Let \( a_{jk} = |a_{jk}| \exp(i\theta_{jk}) \), where \( 0 \leq \theta_{jk} < 2\pi \). We define

\[
\omega_k = [\rho(| A^k |)]^{1/k}, \quad k = 1, 2, \ldots.
\]

**Lemma 1.** If \( k \) and \( r \) are positive integers, then \( \omega_{kr} \leq \omega_k \).

**Proof.** Since \( 0 \leq | A^{kr} | \leq | A^k |^r \), it follows that \( \rho(| A^{kr} |) \leq \rho(| A^k |^r) \). We have always \( \rho(| A^k |^r) = [\rho(| A^k |)]^r \). Consequently,

\[
[\rho(| A^{kr} |)]^{1/k} \leq [\rho(| A^k |)]^{1/k}
\]
or \( \omega_{kr} \leq \omega_k \).

In particular, we deduce

\[
\omega_r \leq \omega_1 = \rho(| A |), \quad r = 1, 2, \ldots.
\]

**Lemma 2.** The \( \omega_r \) \((k = 1, 2, \ldots)\) form a sequence of upper bounds for \( \rho(A) \) which converges to \( \rho(A) \).

**Proof.** Since \( \rho(A^k) \leq \rho(| A^k |) \), it follows that \( \rho(A) \leq [\rho(| A^k |)]^{1/k} = \omega_k \), which proves our first assertion. To prove convergence of the \( \omega_k \) we define the multiplicative matrix norm

\[
N(A) = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} |a_{ij}| \right),
\]

and use the general results [3] that

\[
\lim_{k \to \infty} [N(A^k)]^{1/k} = \rho(A)
\]

and \( [\rho(A)]^k \leq \omega_k^k \leq N(A^k) \). Taking \( k \)th roots we conclude

\[
\lim_{k \to \infty} \omega_k = \rho(A).
\]

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Note. In general the $\omega_k$ do not decrease monotonically to $\rho(A)$. However, Lemma 1 can be used to obtain decreasing subsequences such as $\omega_1, \omega_2, \omega_3, \omega_4, \cdots$.

If $A$ is irreducible, it is known [6] that $\omega_1 = \rho(A)$ if and only if $A = e^{i\Phi}D|A|D^{-1}$, where $D$ is a diagonal matrix whose diagonal entries have modulus unity. If $A$ is of this special form, then $\omega_1 = \omega_k (k = 1, 2, \cdots)$. Furthermore, if we know all the $\omega_k$ are equal, Lemma 2 tells us that $\rho(A)$ has their common value. It is natural to ask what happens in case $\omega_j = \omega_k$ for some $j$ and $k$.

**Theorem 1.** If $A$ has only nonzero entries and if $m > 1$, then $\omega_1 = \omega_m$ if and only if $\rho(A) = \omega_1$.

**Proof.** We have already remarked that $\rho(A) = \omega_1$ implies $\omega_1 = \omega_k (k = 1, 2, \cdots)$ and, in particular, $\omega_1 = \omega_m$.

Conversely, suppose $\omega_1 = \omega_m$ for some $m > 1$. This means

$$\rho(|A|^m) = [\rho(|A|)]^m = \rho(|A|^m).$$

Since $|A|^m$ is a positive matrix and $|A|^m \leq |A|^m$, the Perron-Frobenius theory tells us that $|A|^m = |A|^m$. If we write out the expressions for the $j, k$th entries of $|A|^m$ and $|A|^m$, and use the fact that the modulus of a sum of complex numbers equals the sum of their moduli only when the numbers have the same arguments, we obtain the equation

$$\theta_{ij} + \theta_{kl} + \cdots + \theta_{lm-1k} = \alpha_{jk}.$$

Here, and elsewhere, congruences are modulo $2\pi$; $\alpha_{jk}$ is the argument of the $j, k$th entry of $A^m$ and is independent of the indices $l_1, \cdots, l_{m-1}, 1 \leq l_i \leq n (i = 1, \cdots, m - 1)$. In particular,

$$\alpha_{11} = \theta_{1j} + \theta_{11} + \cdots + \theta_{1l} = \theta_{1j} + \theta_{11} + \cdots + \theta_{1l} + \theta_{ij} = \alpha_{ij}.$$

Similarly,

$$\alpha_{ij} = \theta_{1j} + \theta_{11} + \cdots + \theta_{1l} + \theta_{ij},$$

and

$$\alpha_{jk} = \theta_{j1} + \theta_{11} + \cdots + \theta_{1l} + \theta_{1k}.$$

Therefore,

$$\alpha_{ij} + \alpha_{jk} = \theta_{1j} + \theta_{11} + \cdots + \theta_{1l} + \theta_{1k} + \theta_{j1} + \theta_{j1} + \cdots + \theta_{11} + \theta_{1j}$$

$$= \alpha_{ik} + \alpha_{jj} = \alpha_{ik} + \alpha_{11}.$$

Let $\delta_r = \alpha_{11} - \alpha_{1r}, 1 \leq r \leq n$. Then

$$\alpha_{ik} = \alpha_{ij} + \alpha_{jk} - \alpha_{11}$$

$$= \alpha_{i1} + \alpha_{1k} - \alpha_{11}$$

$$= (2\alpha_{11} - \alpha_{i1}) + \alpha_{ik} - \alpha_{11}$$

$$= \delta_i - \delta_k + \alpha_{11}.$$

Define $D$ to be the matrix

$$\text{diag} (\exp i\delta_1, \cdots, \exp i\delta_n).$$

Then $A^m = (\exp i\alpha_{11}) D|A|^m D^{-1}$ so that

$$\rho(A^m) = \rho(|A|^m).$$
and
\[ \rho(A) = \omega_m = \omega_1. \]

**Theorem 2.** If \( m \) and \( r \) are positive integers with \( r > 1 \), and \( |A^m| > 0 \), then
\[ \omega_m = \omega_{rm} \text{ if and only if } \rho(A) = \omega_m. \]

**Proof.** Suppose \( \omega_m = \omega_{rm} \). Then
\[ [\rho(|A^m|)]^{1/m} = [\rho(|A^{rm}|)]^{1/rm} \]
and
\[ [\rho(|A^m|)]^r = \rho(|A^{rm}|). \]
Since \( |A^m| > 0 \), if we apply Theorem 1 to \( A^m \), we may conclude that
\[ \rho(A^m) = \rho(|A^m|). \]

Hence, \( \rho(A) = \omega_m \).

Conversely, suppose \( \rho(A) = \omega_m \). By Lemma 1, \( \omega_m \geq \omega_{rm} \) and, by Lemma 2, \( \omega_{rm} \geq \rho(A) \). Consequently, \( \omega_m = \omega_{rm} \).

Theorem 1 remains true if we replace the assumption “\( A \) has only nonzero entries” by the slightly weaker condition “for some \( r \) neither the \( r \)th row nor the \( r \)th column of \( A \) has zero entries and \( |A^m| > 0 \).” Theorem 2 can be modified analogously. However, the following example shows that in general it is not possible to relax the assumption of Theorem 1 that \( A \) is a matrix with only nonzero entries to “\( A \) is irreducible.” This relaxation is possible in the Perron-Frobenius theory [6] and one is tempted to try it here. Let
\[
A = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]
Then \( A \) is irreducible but \( \rho(A) = 0 \) and \( \omega_1 = \omega_2 = \sqrt{2} \).

In Theorem 2 we proved that the condition \( \omega_i = \omega_k \), where \( i < k \), \( i \mid k \), and \( |A^k| > 0 \), is sufficient to ensure \( \rho(A) = \omega_i \). One would like to eliminate the requirement \( i \mid k \); however, examples have been constructed showing that, in general, this is not possible.

The following example shows that in some cases a rough estimate for \( \omega_2 \) is a better bound for \( \rho(A) \) than \( \omega_1 \) itself. Let
\[
A = \begin{bmatrix}
2 & -1 \\
1 & -1
\end{bmatrix}.
\]
Then \( \rho(A) \approx 1.62 \), \( \omega_1 \approx 2.62 \) and \( \omega_2 \approx 1.82 \). The square root of the Gerschgorin circle estimated for \( \rho(|A^2|) \) is 2.

**II. Upper Bounds for \( \rho(A) \) in terms of \( \epsilon(A) \).** The Frobenius multiplicative matrix norm \( \epsilon(A) \) [5] is defined by
\[ \epsilon(A) = \left[ \sum_{i,j=1}^{n} |a_{ij}|^2 \right]^{1/2}. \]
Since \( \epsilon \) is a multiplicative norm we have \( \rho(A) \leq \epsilon(A) \). The following result gives the condition for equality.
Lemma 3. The Frobenius norm of $A = [a_{jk}]$ is its spectral radius if and only if $a_{jk} = \epsilon^\theta x_j \bar{x}_k$, where $\bar{x}_k$ denotes the complex conjugate of $x_k$ and $0 \leq \theta < 2\pi$.

Proof. If $a_{jk} = \epsilon^\theta x_j \bar{x}_k$ ($j, k = 1, \cdots, n$), then the only nonzero eigenvalue of $A$ is $\epsilon \left( \sum_{j=1}^n |x_j|^2 \right)$ corresponding to the eigenvector with components $x_j$ ($j = 1, \cdots, n$). Furthermore,

$$[\epsilon(A)]^2 = \sum_{j,k=1}^n |x_j|^2 |x_k|^2 = \left( \sum_{j=1}^n |x_j|^2 \right)^2 = [\rho(A)]^2.$$  

On the other hand, suppose $\rho(A) = \epsilon(A)$. We may assume $\rho(A) > 0$ since $\epsilon(A) = \rho(A) = 0$ implies $A = 0$. Let $\epsilon^\theta \rho(A)$ be an eigenvalue of maximum modulus, whose associated eigenvector has components $x_j$ ($j = 1, \cdots, n$) normalized so that $\rho(A) = \sum_{j=1}^n |x_j|^2$. We have, by the Cauchy-Schwarz inequality,

$$|e^{i\theta} \rho(A)x_j|^2 = \left| \sum_{k=1}^n a_{jk} x_k \right|^2 \leq \left( \sum_{k=1}^n |a_{jk}|^2 \right)\left( \sum_{k=1}^n |x_k|^2 \right), \quad j = 1, \cdots, n.$$  

In order that $\rho(A) = \epsilon(A)$, equality must hold for each $j$ above, which implies

$$a_{jk} = \eta_j \bar{x}_k \quad (j, k = 1, \cdots, n),$$

where the $\eta_j$ are constants. Then

$$e^{i\theta} \rho(A)x_j = \sum_{k=1}^n \eta_j \bar{x}_k x_k = \eta_j \rho(A),$$

so that $\eta_j = e^{i\theta} x_j$ and $a_{jk} = e^{i\theta} x_j \bar{x}_k$, as required.

The following alternate proof of Lemma 3 is due to Alston Householder.

The Frobenius norm is the square root of the sum of the squares of the singular values of $A$, and the largest singular value alone is greater than or equal to the spectral radius. Hence, for equality, the others must be zero implying $A^*A$ is of rank 1. Therefore $A$ is also of rank 1 and hence of the form $ab^*$ where $a$ and $b$ are column vectors. But the only non-null root of $ab^*$ is $b^*a$. From $[\epsilon(ab^*)]^2 = a^*ab^*b = |b^*a|^2$, we conclude $a$ and $b$ are linearly dependent, from which the result follows.

Ideally, one would wish to develop bounds for $\rho(A)$ which depend on $\epsilon(A)$ and some measure of the departure of $A$ from the special form of Lemma 3. One approach is to minimize the Frobenius norm of matrices which are similar to $A$.

Define

$$R_i = \left[ \left( \sum_{j=1}^n |a_{ij}|^2 \right) - |a_{ii}|^2 \right]^{1/2}$$

and

$$C_i = \left[ \left( \sum_{j=1}^n |a_{ji}|^2 \right) - |a_{ii}|^2 \right]^{1/2}.$$  

Theorem 3. If $A$ is an $n \times n$ complex matrix, then

$$[\rho(A)]^2 \leq [\epsilon(A)]^2 - \left[ \max_{1 \leq i \leq n} |R_i - C_i| \right]^2.$$
Proof. We prove the equivalent statement

$$[\rho(A)]^2 \leq [\epsilon(A)]^2 - (R_i - C_i)^2, \quad i = 1, \ldots, n.$$  

Suppose first that neither \( R_i \) nor \( C_i \) is zero. Let \( D_v \) be the diagonal matrix whose diagonal entries are all unity except for \( v \neq 0 \) in the \( i \)th position. Then \( \rho(D_vAD_v^{-1}) = \rho(A) \). Hence, \( [\rho(A)]^2 \leq [\epsilon(D_vAD_v^{-1})]^2 = [\epsilon(A)]^2 - R_i^2 - C_i^2 + v^2R_i^2 + v^{-2}C_i^2 \). If we minimize the right-hand expression over \( v \) we obtain \( v^2 = C_i/R_i \), and

$$[\rho(A)]^2 \leq [\epsilon(A)]^2 - (R_i - C_i)^2.$$  

Since \( \rho(A) \), \( \epsilon(A) \), \( R_i \) and \( C_i \) all depend continuously on the entries of \( A \), it follows that the restriction \( R_i, C_i \neq 0 \) can be removed.

If it happens that \( R_i, C_i \neq 0 \), where \( i \) is the index which gives the maximum in Theorem 3, then Theorem 3 may be applied to the matrix \( D_vAD_v^{-1} \), where \( v^2 = C_i/R_i \), giving a possible improvement in the bound for \( \rho(A) \).

If \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \), then it is easily seen that

$$\inf \{[\epsilon(SAS^{-1})]^2 : S \text{ nonsingular}\} = \sum_{i=1}^{n} |\lambda_i|^2.$$  

Hence, the bound given by Theorem 3 must be greater than or equal to \( \sum_{i=1}^{n} |\lambda_i|^2 \).

We will now consider bounds which in some cases are actually less than \( \sum_{i=1}^{n} |\lambda_i|^2 \). Let \( \text{tr} \ A \) be the trace of \( A \).

**Theorem 4.** If \( A \) is an \( n \times n \) complex matrix, then

$$\rho(A) \leq (1 - 1/n)^{1/2}[[\epsilon(SAS^{-1})]^2 - ||\text{tr} \ A||^2/n]^{1/2} + ||\text{tr} \ A||/n,$$

for any nonsingular \( S \).

**Proof.** Let \( \lambda_M \) be an eigenvalue of maximum modulus. Then, from

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq [\epsilon(SAS^{-1})]^2$$  

by an application of the Cauchy-Schwarz inequality we find

$$|\lambda_M|^2 \leq [\epsilon(SAS^{-1})]^2 - \sum_{i \neq M} |\lambda_i|^2$$

$$\leq [\epsilon(SAS^{-1})]^2 - \sum_{i \neq M} |\lambda_i|^2/(n - 1)$$

$$= [\epsilon(SAS^{-1})]^2 - ||\text{tr} \ A - \lambda_M||^2/(n - 1),$$

from which it follows, by elementary means, that

$$|\lambda_M| \leq (1 - 1/n)^{1/2}[[\epsilon(SAS^{-1})]^2 - ||\text{tr} \ A||^2/n]^{1/2} + ||\text{tr} \ A||/n.$$  

**Theorem 5.** Let \( A \) be an \( n \times n \) complex nonsingular matrix. Then

$$[\rho(A)]^2 \leq [\epsilon(SAS^{-1})]^2 - (n - 1)||\det A||^2/[\epsilon(SAS^{-1})]^2]^{1/(n-1)}$$

for any nonsingular \( S \).

**Proof.** Let \( \lambda_M \) be an eigenvalue of maximum modulus. As in Theorem 4,

$$|\lambda_M|^2 \leq [\epsilon(SAS^{-1})]^2 - \sum_{i \neq M} |\lambda_i|^2.$$  

An application of the arithmetic-geometric mean inequality yields

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\[ |\lambda_M|^2 \leq [\epsilon(SAS^{-1})]^2 - (n - 1) \prod_{i \neq M} |\lambda_i|^{2/(n-1)}. \]

But
\[ \prod_{i \neq M} |\lambda_i|^{2/(n-1)} = \left( |\det A|^{2} / |\lambda_M|^2 \right)^{1/(n-1)} \]
\[ \geq \left| \det A \right|^{2} / [\epsilon(SAS^{-1})]^2 \right|^{1/(n-1)}, \]
from which the result follows.

We observe that the quantity \([\epsilon(SAS^{-1})]^2\) occurring in Theorems 4 and 5 may be replaced by the bound for it given by Theorem 3. We use this fact in the discussion of the following numerical example which illustrates the various bounds. Let
\[ A = \begin{bmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}. \]

Then \(\rho(A) = 11\) and \(\left( \sum_{i=1}^{3} |\lambda_i|^2 \right)^{1/2} = 11.58\). The Ledermann bound [2] is 16.77.

The bound of Theorem 3 is 11.9 and, using this bound, we obtain from Theorem 4 the bound 11.3 and from Theorem 5 the bound 11.6.

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