Mantissa Distributions

By Alan G. Konheim

Let $b$ be an integer, at least 2, and write each positive real number in the form

$$(1) \quad x = mb^c,$$

where $m$ (the mantissa) satisfies $1/b \leq m < 1$ and $c$ (the characteristic) is an integer.

We define the product of mantissas* $m_1$ and $m_2$ by

$$(2) \quad m_1 \cdot m_2 = \begin{cases} 
  m_1m_2 & \text{if } 1/b \leq m_1m_2 < 1, \\
  bm_1m_2 & \text{if } 1/b^2 \leq m_1m_2 < 1/b. 
\end{cases}$$

Now suppose that $M_1$ and $M_2$ are independent, identically distributed random variables, each taking on values in the interval $[1/b, 1)$ such that

$$(3) \quad \Pr(M_1 \cdot M_2 \leq x) = \Pr(M \leq x).$$

What are all of the possible choices for the distribution function of $M_1$? The answer is provided by the following

**Theorem.** $\Pr(M_1 \leq x) = F_n(x)$ or $F_\infty(x)$ ($n = 1, 2, \ldots$), where

$$F_n(x) = \begin{cases} 
  0 & \text{if } -\infty < x < b^{-1}, \\
  1/n & \text{if } b^{-1} \leq x < b^{-1+(1/n)}, \\
  2/n & \text{if } b^{-1+(1/n)} \leq x < b^{-1+(2/n)}, \\
  \vdots \\
  1 & \text{if } b^{-1} \leq x < \infty, \\
\end{cases}$$

and

$$F_\infty(x) = \begin{cases} 
  0 & \text{if } -\infty < x < b^{-1}, \\
  1 + 1/n \left[ \frac{\log x}{\log b} + 1 \right] & \text{if } b^{-1} \leq x < 1, \\
  1 & \text{if } 1 \leq x < \infty, \\
\end{cases}$$

**Proof.** We will write $M_i = b^{-\Theta_i}$ ($i = 1, 2$), where $\Theta_1$ and $\Theta_2$ are independent, identically distributed random variables, taking on values in $(0, 1]$. Note that

$$M_1 \cdot M_2 = b^{-(\Theta_1 + \Theta_2)}.$$

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* If $m_i$ is the mantissa of $x_i$ then $m_1 \cdot m_2$ is the mantissa of $x_1x_2$.

† [ ] denotes ‘the integer part of.’
where * denotes addition modulo one. Thus (3) is equivalent to requiring that \( \Theta_1 + \Theta_2 \) and \( \Theta_1 \) have the same distribution. If we set

\[
\phi(n) = E\{e^{2\pi i \Theta_1}\} = \int_0^1 e^{2\pi i \theta_1} \, dF_{\Theta_1}(\theta_1),
\]

then (3) and the independence of \( \Theta_1, \Theta_2 \) imply

\[
\phi(n) = E\{e^{2\pi i (\Theta_1 + \Theta_2)}\} = E\{e^{2\pi i (\Theta_1 + \Theta_2)}\} = \phi^2(n)
\]

so that \( \phi(n) = 0 \) or 1. Certainly \( \phi(0) = 1 \). There are two cases to be examined.

Case 1. \( \phi(n) = 0 \) for all \( n \neq 0 \).

It follows from the uniqueness theorem for Fourier-Stieltjes series that \( dF_{\Theta_1}(d\theta_1) = d\theta_1 \) and hence \( \Pr(M_1 \leq x) = F_\alpha(x) \).

Case 2. \( \phi(n) = 1 \) for some \( n \neq 0 \).

Let \( m \) be the smallest positive integer such that \( \phi(m) = 1 \). Then

\[
0 = \int_0^1 (1 - e^{2\pi i \theta_1}) \, dF_{\Theta_1}(\theta_1) = \int_0^1 (1 - \cos 2\pi m\theta_1) \, dF_{\Theta_1}(\theta_1).
\]

It follows that \( F_{\Theta_1} \) is a 'step function' with points of discontinuity at \( \theta_k = k/m \) \( (k = 1, 2, \ldots, m) \) and, hence, \( \phi(n + m) = \phi(n) \) \( (n = 0, \pm 1, \pm 2, \ldots) \). We assert that \( \phi(n) = 1 \) if and only if \( n = km \) for some integer \( k \); for if \( \phi(n) = 1 \) with \( km < n < (k + 1)m \) then \( \phi(n - km) = \phi(n) = 1 \) while \( 0 < n - km < m \) contradicting the minimality of \( m \). The uniqueness theorem for Fourier-Stieltjes series now shows that \( \Pr(M_1 \leq x) = F_m(x) \).

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**New Primes of the Form \( n^4 + 1 \)**

**By A. Gloden**

This note presents some results of a continuation of the author's systematic factorization of integers of the form \( n^4 + 1 \) [1].

An electronic computer at l'Institut Blaise Pascal in Paris has been used to find solutions of the congruence \( x^4 + 1 \equiv 0 \pmod{p} \) for all primes of the form \( 8k + 1 \) in the interval \( 10^6 < p < 4 \cdot 10^6 \), thereby extending the previous range of such tables listed in [1].

With the aid of these tables, the complete factorization of \( n^4 + 1 \) has now been effected for all even values of \( n \) less than 2040 and for all odd values less than 2397.

Thus, the primality of \( \frac{1}{2}(n^4 + 1) \) has been established for the following 116 values of \( n \):

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