β may be obtained from Gershgorin's theorem. A method of obtaining lower bounds for the least positive eigenvalue of a certain type matrix is discussed in [5].

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1. G. Birkhoff, R. S. Varga & D. Young, *Alternating Direction Implicit Methods*, Advances in Computers, Vol. 3, Academic Press, New York, 1962.
2. C. de Boor & J. R. Rice, "Chebyshev approximation by a $\Pi(x - r_i)/(x + s_i)$ and application to ADI iteration," *J. Soc. Indust. Appl. Math.*, v. 11, 1963, pp. 159–169. MR **28** #466.
3. J. Douglas, "Alternating direction methods for three space variables," *Numer. Math.*, v. 4, 1962, pp. 41–63. MR **24** #B2122.
4. J. Douglas & C. Pearcy, "On convergence of alternating direction procedures in the presence of singular operators," *Numer. Math.*, v. 5, 1963, pp. 175–184. MR **27** #4384.
5. W. H. Guilinger & R. B. Kellogg, "Eigenvalue lower bounds," submitted to the *Bettis Technical Review*.
6. D. W. Peaceman & H. H. Rachford, Jr., "The numerical solution of parabolic and elliptic differential equations," *J. Soc. Indust. Appl. Math.*, v. 3, 1955, pp. 28–41. MR **17**, 196.
7. R. B. Smith & J. Spanier, *HOT-1: A Two Dimensional Steady-State Heat Conduction Program for the Philco-2000*, WAPD-TM-465, Bettis Atomic Power Laboratory, Pittsburgh, Pa., 1964.
8. J. Spanier & W. H. Guilinger, "Matrix conditioning for alternating direction methods," *Bettis Technical Review*, WAPD-BT-29, 1963, pp. 69–79.
9. R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N. J., 1962. MR **28** #1725.
10. E. L. Wachspress, "Optimum alternating-direction-implicit iteration parameters for a model problem," *J. Soc. Indust. Appl. Math.*, v. 10, 1962, pp. 339–350. MR **27** #921.
11. E. L. Wachspress, "Extended application of alternating direction implicit iteration model problem theory," *J. Soc. Indust. Appl. Math.*, v. 11, 1963, pp. 994–1016.
12. E. L. Wachspress & G. J. Habetler, "An alternating direction implicit iteration technique," *J. Soc. Indust. Appl. Math.*, v. 8, 1960, pp. 403–424. MR **22** #5132.

An Iterative Method for Computing the Generalized Inverse of an Arbitrary Matrix

By Adi Ben-Israel

Abstract. The iterative process, $X_{n+1} = X_n(2I - AX_n)$, for computing A^{-1}, is generalized to obtain the generalized inverse.

An iterative method for inverting a matrix, due to Schulz [1], is based on the convergence of the sequence of matrices, defined recursively by

$$X_{n+1} = X_n(2I - AX_n) \qquad (n = 0, 1, \cdots) \tag{1}$$

to the inverse A^{-1} of A, whenever X_0 approximates A^{-1}. In this note the process (1) is generalized to yield a sequence of matrices converging to A^+, the generalized inverse of A [2].

Let A denote an $m \times n$ complex matrix, A^* its conjugate transpose, $P_{R(A)}$ the perpendicular projection of E^m on the range of A, $P_{R(A^*)}$ the perpendicular projection of E^n on the range of A^*, and A^+ the generalized inverse of A.

Theorem. *The sequence of matrices defined by*

$$X_{n+1} = X_n(2P_{R(A)} - AX_n) \qquad (n = 0, 1, \cdots), \tag{2}$$

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where X_0 is an $n \times m$ complex matrix satisfying

$$X_0 = A^*B_0 \quad \text{for some nonsingular } m \times m \text{ matrix } B_0, \tag{3}$$

$$X_0 = C_0A^* \quad \text{for some nonsingular } n \times n \text{ matrix } C_0, \tag{4}$$

$$\| AX_0 - P_{R(A)}\| < 1, \tag{5}$$

$$\| X_0A - P_{R(A^*)}\| < 1, \tag{6}$$

converges to the generalized inverse A^+ of A.[1]

Proof. As in [3], the generalized inverse A^+ is characterized as the unique solution of the matrix equations,

$$AX = P_{R(A)}, \tag{7}$$

$$XA = P_{R(A^*)}. \tag{8}$$

Thus it suffices to prove that the sequence (2) satisfies:

$$\lim_{n\to\infty} \| AX_n - P_{R(A)}\| = 0, \tag{9}$$

$$\lim_{n\to\infty} \| X_nA - P_{R(A^*)}\| = 0. \tag{10}$$

First we verify from (2), (3), (4) that

$$X_n = A^*B_n \tag{11}$$

$$X_n = C_nA^* \qquad (n = 0, 1, \cdots) \tag{12}$$

(where B_n, C_n are recursively computed as

$$B_{n+1} = B_n(2P_{R(A)} - AA^*B_n),$$

$$C_{n+1} = C_n(2P_{R(A^*)} - A^*AC_n),$$

but are not used in the sequel).

Now, from (2),

$$P_{R(A)} - AX_{n+1} = (P_{R(A)} - AX_n)P_{R(A)} - AX_n(P_{R(A)} - AX_n); \tag{13}$$

using (12), it follows that $AX_nP_{R(A)} = P_{R(A)}AX_n$.

Therefore

$$P_{R(A)} - AX_{n+1} = (P_{R(A)} - AX_n)^2$$

and

$$\| P_{R(A)} - AX_{n+1} \| \leqq \| P_{R(A)} - AX_n \|^2 \qquad (n = 0, 1, \cdots), \tag{14}$$

which, by (5), proves (9).

To prove (10) we write

$$P_{R(A^*)} - X_{n+1}A = P_{R(A^*)} - X_n(2P_{R(A)} - AX_n)A,$$

which is rewritten, by (11), as

$$P_{R(A^*)} - X_{n+1}A = P_{R(A^*)} - P_{R(A^*)}X_nA - X_nA + (X_nA)^2 = (P_{R(A^*)} - X_nA)^2.$$

[1] $\| \;\; \|$ is a multiplicative matrix norm.

Thus

$$\| P_{R(A^*)} - X_{n+1}A \| \leqq \| P_{R(A^*)} - X_nA \|^2 \qquad (n = 0, 1, \cdots) \tag{15}$$

which, by (6), proves (10).

Remarks. (i) Similarly, the sequence defined by

$$X_{n+1} = (2P_{R(A^*)} - X_nA)X_n \qquad (n = 0, 1, \cdots), \tag{16}$$

with X_0 satisfying (3), (4), (5), (6), converges to A^+.

(ii) When A is nonsingular, both (2) and (16) reduce to the well-known process (1) due to Schulz [1], further studied by Dück in [4].

(iii) Conditions (5), (6) can not be weakened as shown by:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad P_{R(A)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and, taking

$$X_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

which satisfies (3), (4) but $\| AX_0 - P_{R(A)} \| = 1$ under the sum-of-squares norm.

(iv) The practical significance of the process proposed here is impaired by the need for knowledge of $P_{R(A)}$. In fact, the direct computation of A^+ requires little more than the computation of $P_{R(A)}$ and of $P_{R(A^*)}$, and not substantially more than the computation of one alone. For any matrix A can be expressed in the form $A = FR^*$ where the columns of F are linearly independent as are those of R. Then, as shown by Householder in [5],

$$P_{R(A)} = F(F^*F)^{-1}F^*$$

and

$$P_{R(A^*)} = R(R^*R)^{-1}R^*,$$

whereas

$$A^+ = R(R^*R)^{-1}(F^*F)^{-1}F^*.$$

While only one of the projections $P_{R(A)}$, $P_{R(A^*)}$ is needed for the computation by the method proposed here, both are needed for testing (5) and (6).

(v) In the case where A is of full rank, the method proposed here is applicable. For, if rank $A = m$, $P_{R(A)} = I_{m\times m}$ and (2) reads:

$$X_{n+1} = X_n(2I - AX_n). \tag{17}$$

In this case, $A^+ = A^*(AA^*)^{-1}$ and, indeed, by (11), we verify that $X_n = A^*B_n$, where B_n converges to $(AA^*)^{-1}$.

Similarly, if rank $A = n$, $P_{R(A^*)} = I_{n\times n}$ and (16) becomes

$$X_{n+1} = (2I - X_nA)X_n. \tag{18}$$

Example. Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

and take

$$X_0 = \frac{1}{2} A^* = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Here, formula (17) is used to obtain:

$$X_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \left\{ 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

$$= \frac{1}{4} \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix},$$

$$X_2 = \frac{1}{16} \begin{pmatrix} 10 & 5 \\ 5 & 10 \\ -5 & 5 \end{pmatrix},$$

$$X_3 = \frac{1}{256} \begin{pmatrix} 170 & 85 \\ 85 & 170 \\ -85 & 85 \end{pmatrix}, \qquad \text{etc.,}$$

converging to:

$$A^+ = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

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1. G. Schulz, "Iterative Berechnung der reziproken Matrix," *Z. Angew. Math. Mech.* v. 13, 1933, pp. 57–59.
2. R. Penrose, "A generalized inverse for matrices," *Proc. Cambridge Philos. Soc.*, v. 51, 1955, pp. 406–413. MR **16**, 1082.
3. A. Ben-Israel & A. Charnes, "Contributions to the theory of generalized inverses," *J. Soc. Indust. Appl. Math.*, v. 11, 1963, pp. 667–699.
4. W. Dück, "Fehlerabschätzungen für das Iterationsverfahren von Schulz zur Bestimmung der Inversen einer Matrix," *Z. Angew. Math. Mech.*, v. 40, 1960, pp. 192–194. MR **22** #3102.
5. A. S. Householder, *Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964.

A Note on the Maximum Value of Determinants over the Complex Field

By C. H. Yang

The purpose of this note is to extend a theorem on determinants over the real field to the corresponding theorem over the complex field.

Theorem. *Let $D(n)$ be an nth order determinant with complex numbers as its entries. Then*

$$\max_{|a_{jk}| \leq K} |D(n)| = \max_{|a_{jk}| = K} |D(n)|. \tag{1}$$

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