problem. Furthermore, the programming involved is simple enough to be assigned as a laboratory exercise in a numerical analysis course. As an example, quadrature formulae adapted to a logarithmically singular kernel are given.

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Quadrature Formulas Using Derivatives

By Lawrence F. Shampine

For k odd, we shall derive a new quadrature formula of the type

$$\int_{-1}^{1} f(x)\,dx \cong 2 \sum_{j=0}^{(k-1)/2} \frac{f^{(2j)}(0)}{(2j+1)!}[1 - c_j] + \sum_{l=1}^{m} a_l[f(x_l) + f(-x_l)],$$

which is exact for all polynomials of degree up to $4m + k - 2$. A similar formula holds for k even. The formulas closely resemble those of Hammer and Wicke [1]: for k odd,

$$\int_{-1}^{1} f(x)\,dx \cong 2 \sum_{j=0}^{(k-1)/2} \frac{f^{(2j)}(0)}{(2j+1)!} + \sum_{l=1}^{m} a_l[f^{(k)}(x_l) + f^{(k)}(-x_l)],$$

and a similar formula for k even. Their formulas require the use of nonclassical orthogonal polynomials. The formulas stated above are derived very simply with the use of Jacobi polynomials and would, presumably, be useful in situations similar to those envisioned by Hammer and Wicke.

$f(x)$ can be split into even and odd parts. The form of the formula is such as to integrate the odd part exactly. Let us write $f(x)$ in the form

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(2j)}(0)}{(2j)!} x^{2j},$$

Received December 3, 1964. This work was supported in part by the National Science Foundation and in part by the United States Atomic Energy Commission. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

which is certainly valid for polynomials, and let us define the polynomial $P(x)$ as the terms through degree $k - 1$ of the series.

$$\int_{-1}^{1} f(x)\, dx = 2 \sum_{j=1}^{(k-1)/2} \frac{f^{(2j)}(0)}{(2j+1)!} + \int_{-1}^{1} \sum_{(k+1)/2}^{\infty} \frac{f^{(2j)}(0)}{(2j)!} x^{2j}\, dx.$$

This last integral may be written as

$$I = \int_{-1}^{1} x^{k+1} G(x)\, dx = 2 \int_{0}^{1} u^{(k+1)/2} F(u)\, du$$

$$= 2^{-(k+1)/2} \int_{-1}^{1} (1 + y)^{(k+1)/2} F(y)\, dy,$$

where G is a polynomial in x^2. We evaluate this by an m-point formula of highest degree of precision. It is of Gauss-Jacobi type and it is known [2, p. 111 ff.] that

$$\int_{-1}^{1} (1 + y)^{(k+1)/2} F(y)\, dy \cong \sum_{l=1}^{m} b_l F(y_l),$$

where y_l are the roots of the Jacobi polynomial $P_m^{(0,(k+1)/2)}(y)$; the b_l are found, as usual, from the polynomials. $x_l^2 = u_l = \frac{1}{2}(1 + y_l)$.

$$I = 2 \sum_{l=1}^{m} b_l F(u_l) = \sum_{l=1}^{m} b_l [G(x_l) + G(-x_l)].$$

Now

$$x^{k+1} G(x) = f(x) - P(x).$$

Let

$$a_l = \frac{b_l}{x_l^{k+1}}, \quad \text{noting that no} \quad x_l = 0.$$

Then

$$I = \sum_{l=1}^{m} a_l [f(x_l) + f(-x_l)] - 2 \sum_{l=1}^{m} a_l P(x_l).$$

$$2 \sum_{l=1}^{m} a_l P(x_l) = 2 \sum_{l=1}^{m} a_l \sum_{j=0}^{(k-1)/2} \frac{f^{(2j)}(0)}{(2j)!} x_l^{2j}.$$

With the definition

$$c_j = (2j + 1) \sum_{l=1}^{m} b_l x_l^{2j-k-1}, \qquad j = 0, 1, \cdots, \frac{k-1}{2},$$

we obtain the formula stated.

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