and take

$$X_0 = \frac{1}{2}A^* = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Here, formula (17) is used to obtain:

$$X_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}\left\{2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}\right\}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix},$$

$$X_2 = \frac{1}{16}\begin{pmatrix} 10 & 5 \\ 5 & 10 \\ -5 & 5 \end{pmatrix},$$

$$X_3 = \frac{1}{256}\begin{pmatrix} 170 & 85 \\ 85 & 170 \\ -85 & 85 \end{pmatrix}, \qquad \text{etc.},$$

converging to:

$$A^+ = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

Technion, Israel Institute of Technology
Haifa, Israel

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A Note on the Maximum Value of Determinants over the Complex Field

By C. H. Yang

The purpose of this note is to extend a theorem on determinants over the real field to the corresponding theorem over the complex field.

Theorem. *Let $D(n)$ be an nth order determinant with complex numbers as its entries. Then*

$$(1) \qquad \operatorname*{Max}_{|a_{jk}|\leq K} |D(n)| = \operatorname*{Max}_{|a_{jk}|=K} |D(n)|.$$

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In other words, $D(n)$ is a function of n^2 variables a_{jk} which vary over the bounded and closed domain $\bar{D}$: $\{|a_{jk}| \leqq K\}$; hence this function is bounded and attains its maximum value on the boundary of the domain $\bar{D}$.

Proof. Let $a_{jk} = r_{jk}e^{i\theta_{jk}}$ and $A_{jk} = R_{jk}e^{i\phi_{jk}}$ = the co-factor of a_{jk}, where $K \geqq r_{jk} \geqq 0$ and $R_{jk} \geqq 0$. Then, expanding by the jth row, we have

$$(2) \qquad \begin{aligned} |D(n)| &= \left| \sum_{k=1}^{n} a_{jk}A_{jk} \right| = \left| \sum_{k=1}^{n} r_{jk}R_{jk}\, e^{i(\theta_{jk}+\phi_{jk})} \right| \\ &\leqq \sum_{k=1}^{n} r_{jk}R_{jk} \leqq \sum_{k=1}^{n} KR_{jk} = D'(n), \end{aligned}$$

where $D'(n)$ is the nth order determinant whose entries are

$$(3) \qquad a'_{jk} = \begin{cases} a_{jk}, & \text{if } r_{jk} = K \text{ and } \theta_{jk} + \phi_{jk} \equiv 0 \pmod{2\pi}, \\ Ke^{-i\phi_{jk}}, & \text{if } r_{jk} < K \text{ or } \theta_{jk} + \phi_{jk} \not\equiv 0 \pmod{2\pi}. \end{cases}$$

By applying the same process to the other rows, we obtain a determinant $D^*(n)$ whose entries $|a^*_{jk}| = K$ and $|D^*(n)| \geqq |D(n)|$. Hence, $\text{Max}_{|a_{jk}|\leqq K} |D(n)| \leqq \text{Max}_{|a_{jk}|=K} |D(n)|$; thus the proof of the theorem can be completed since the reverse inequality is trivial.

50 Mohigan Drive
Oneonta, New York

On the Numerical Solution of $y' = f(x, y)$ by a Class of Formulae Based on Rational Approximation

By John D. Lambert and Brian Shaw

1. Introduction. Most finite difference formulae in common usage for the numerical solution of first-order differential equations are based on polynomial approximation. Two exceptions are the formulae based on exponential approximation proposed by Brock and Murray [1], and the formulae of Gautschi [2] which are derived from trigonometric polynomials. The use of rational functions as approximants has been studied by many authors, including Remes [3], Maehly [4] and Stoer [5], but the main concern of most of this work has been the direct approximation of a given function. Algorithms for interpolation based on rational functions have been proposed by Wynn [6], and methods for numerical integration and differentiation based on Padé approximation have been studied by Kopal [7]. It is the purpose of the present paper to derive a class of formulae, based on rational approximation, for the numerical solution of the initial value problem

$$(1) \qquad y' = f(x, y), \qquad y(x_0) = y_0 .$$

The formulae proposed give exact results when the theoretical solution of (1) is a rational function of a certain degree, just as many of the classical difference formulae

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