

# Evaluation at Half Periods of Weierstrass' Elliptic Functions with Double Periods 1 and $e^{i\alpha}$

By Chih-Bing Ling

In two previous papers [1], [2], the following three Weierstrass' elliptic functions at half periods were evaluated to 16D for the cases of rectangular and rhombic primitive period-parallelograms with double periods  $2\omega_1 = 1$  and  $2\omega_2 = ai$  and  $\frac{1}{2} + ci$ , respectively:

$$(1) \quad e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_2), \quad e_3 = \wp(\omega_3),$$

where  $\omega_3$  is defined by

$$(2) \quad \omega_1 + \omega_2 + \omega_3 = 0.$$

In addition, two related coefficients  $\sigma_4$  and  $\sigma_6$  were also evaluated to 16D. The coefficient  $\sigma_{2s}$  for  $s \geq 2$  is defined by the double series

$$(3) \quad \sigma_{2s}(2\omega_1, 2\omega_2) = \sum'_{m,n=-\infty}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_2)^{2s}},$$

where the accent on the summation sign denotes the omission of simultaneous zeros of  $m$  and  $n$ . The coefficient  $\sigma_{2s}$  in these two cases has been further evaluated up to  $2s = 50$ , also to 16D. The values up to  $2s = 20$  were tabulated in a recent paper [3].

In this paper, the evaluation is extended to the case in which the double periods are  $2\omega_1 = 1$  and  $2\omega_2 = e^{i\alpha}$ . It is not difficult to show that

$$(4) \quad \sigma_{2s}(1, e^{i\alpha}) = (1 + e^{i\alpha})^{-2s} \sigma_{2s}(1, \frac{1}{2} + ci) \quad (s \geq 2)$$

and similarly,

$$(5) \quad \begin{aligned} e_1(1, e^{i\alpha}) &= (1 + e^{i\alpha})^{-2} e_3(1, \frac{1}{2} + ci), \\ e_2(1, e^{i\alpha}) &= (1 + e^{i\alpha})^{-2} e_2(1, \frac{1}{2} + ci), \\ e_3(1, e^{i\alpha}) &= (1 + e^{i\alpha})^{-2} e_1(1, \frac{1}{2} + ci), \end{aligned}$$

where

$$(6) \quad c = \frac{1}{2} \tan \frac{\alpha}{2}.$$

It is thus seen that, apart from a complex multiplier,  $\sigma_{2s}$  and  $e_r$  can be evaluated by using the same method as in the case of rhombic primitive period-parallelogram.

Furthermore, it can be shown that when  $\alpha$  and  $\beta$  are supplementary angles,

$$(7) \quad \sigma_{2s}(1, e^{i\beta}) = e^{2si\alpha} \sigma_{2s}(1, e^{i\alpha}) \quad (s \geq 2)$$

and

$$(8) \quad \begin{aligned} e_1(1, e^{i\beta}) &= e^{2i\alpha} e_2(1, e^{i\alpha}), \\ e_2(1, e^{i\beta}) &= e^{2i\alpha} e_1(1, e^{i\alpha}), \\ e_3(1, e^{i\beta}) &= e^{2i\alpha} e_3(1, e^{i\alpha}). \end{aligned}$$

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TABLE 1

$\alpha^\circ$	$\sigma_4$	$\sigma_6$
10 } 170 }	2.20327 53592 69607 (3) $\mp 8.01926 64866 63102 (2)i$	-6.28162 53571 93760 (4) $\pm 3.62669 80909 24197 (4)i$
20 } 160 }	1.13982 48849 40138 (2) $\mp 9.56426 64055 99553 (1)i$	-5.79789 14112 90271 (2) $\pm 1.00422 42501 12197 (3)i$
30 } 150 }	1.50455 42089 69734 (1) $\mp 2.60596 43326 77182 (1)i$	0 $\pm 1.06195 78899 46210 (2)i$
40 } 140 }	1.64345 02330 26281 $\mp 9.32046 94280 08436 i$	1.08183 16750 82049 (1) $\pm 1.87378 74264 79455 (1)i$
50 } 130 }	-5.29056 91229 52277 (-1) $\mp 3.00043 08482 43006 i$	7.60880 13435 54263 $\pm 4.39294 35039 11440 i$
60 } 120 }	0	5.86303 16934 25402
70 } 110 }	1.37646 54366 09637 $\pm 1.15499 16401 87420 i$	3.65983 74207 88683 $\mp 2.11300 81200 82612 i$
80 } 100 }	2.64581 45492 33692 $\pm 9.62997 74130 95135 (-1)i$	1.14643 33070 49294 $\mp 1.98568 07352 98588 i$
90	3.15121 20021 53898	0

TABLE 2

$\alpha^\circ$	$e_1$	$e_2$	$e_3$
10 } 170 }	1.06629 64687 30742 (2) $\mp 1.88016 41620 31514 (1)i$	1.06629 63248 53731 (2) $\mp 1.88017 23217 02293 (1)i$	-2.13259 27935 84473 (2) $\pm 3.76033 64837 33807 (1)i$
20 } 160 }	2.56611 68976 91807 (1) $\mp 9.24568 62556 15094 i$	2.56006 08466 86371 (1) $\mp 9.41207 48895 00090 i$	-5.12617 77443 78178 (1) $\pm 1.86577 61145 11518 (1)i$
30 } 150 }	1.10500 69968 70304 (1) $\mp 5.41185 82773 81614 i$	1.02118 41734 24509 (1) $\mp 6.86371 21678 01545 i$	-2.12619 11702 94813 (1) $\pm 1.22755 70445 18316 (1)i$
40 } 140 }	6.81070 58217 30424 $\mp 2.77473 54068 39656 i$	3.91524 75957 82218 $\mp 6.22540 81498 19888 i$	-1.07259 53417 51264 (1) $\pm 9.00014 35566 59544 i$
50 } 130 }	5.77763 72395 07811 $\mp 9.92377 51947 74830 (-1)i$	-2.59751 02764 67654 (-2) $\mp 5.86218 64953 74237 i$	-5.75166 21367 43134 $\pm 6.85456 40148 51720 i$
60 } 120 }	5.89834 39694 84770	-2.94917 19847 42385 $\mp 5.10811 57178 32557 i$	-2.94917 19847 42385 $\pm 5.10811 57178 32557 i$
70 } 110 }	6.34386 76730 64185 $\pm 3.59666 66987 31461 (-1)i$	-5.09087 38578 44618 $\mp 3.80223 88839 05204 i$	-1.25299 38152 19567 $\pm 3.44257 22140 32058 i$
80 } 100 }	6.72908 70025 06195 $\pm 2.88354 16209 99233 (-1)i$	-6.42189 63327 31407 $\mp 2.03051 90227 49862 i$	-3.07190 66977 47873 (-1) $\pm 1.74216 48606 49938 i$
90	6.87518 58180 20373	-6.87518 58180 20373	0

TABLE 3

$\theta^\circ$	tan $\theta$					cot $\theta$						
1	0.01745	50649	28217	58576	51289	0	57.28996	16307	59424	68727	815	89
2	0.03492	07694	91747	73050	04026	3	28.63625	32829	15603	55075	651	88
3	0.05240	77792	83041	20403	88058	2	19.08113	66877	28211	06340	675	87
4	0.06992	68119	43510	41366	69210	6	14.30066	62567	11927	91012	805	86
5	0.08748	86635	25924	00522	20186	7	11.43005	23027	61343	06721	086	85
6	0.10510	42352	65676	46251	15024		9.51436	44542	22584	92968	3971	84
7	0.12278	45609	02904	59113	42311		8.14434	64279	74594	02382	5661	83
8	0.14054	08347	02391	44683	81177		7.11536	97223	84208	74823	0566	82
9	0.15838	44403	24536	29383	88831		6.31375	15146	75043	09897	9464	81
10	0.17632	69807	08464	97347	10904		5.67128	18196	17709	53099	4418	80
11	0.19438	03091	37718	48424	31942		5.14455	40159	70310	13472	3221	79
12	0.21255	65616	70022	12525	95917		4.70463	01094	78454	23358	6235	78
13	0.23086	81911	25563	11174	81456		4.33147	58742	84155	54554	6168	77
14	0.24932	80028	43180	69162	40399		4.01078	09335	35844	71634	5715	76
15	0.26794	91924	31122	70647	25537		3.73205	08075	68877	29352	7446	75
16	0.28674	53857	58807	94004	27581		3.48741	44438	40908	65069	6224	74
17	0.30573	06814	58660	35573	45420		3.27085	26184	84140	86530	8856	73
18	0.32491	96962	32906	32615	58714		3.07768	35371	75253	40257	0291	72
19	0.34432	76132	89665	24195	72658		2.90421	08776	75822	80257	9326	71
20	0.36397	02342	66202	36135	10479		2.74747	74194	54622	27876	1664	70
21	0.38386	40350	35415	79597	14484		2.60508	90646	93801	53625	8412	69
22	0.40402	62258	35156	81132	23481		2.47508	68534	16295	82524	0013	68
23	0.42447	48162	09604	74202	35321		2.35585	23658	23752	83393	9587	67
24	0.44522	86853	08536	16392	23670		2.24603	67739	04216	05416	3321	66
25	0.46630	76581	54998	59283	00062		2.14450	69205	09558	61635	6261	65
26	0.48773	25885	65861	42277	31111		2.05030	38415	79296	21689	9011	64
27	0.50952	54494	94428	81051	37069		1.96261	05055	05150	58230	4640	63
28	0.53170	94316	61478	74807	59159		1.88072	64653	46332	01236	0838	62
29	0.55430	90514	52768	91782	07631		1.80404	77552	71423	93738	1785	61
30	0.57735	02691	89625	76450	91488		1.73205	08075	68877	29352	7446	60
31	0.60086	06190	27560	41487	86644		1.66427	94823	50517	91103	0496	59
32	0.62486	93519	09327	50978	05108		1.60033	45290	41050	35532	6733	58
33	0.64940	75931	97510	57698	20629		1.53986	49638	14582	90482	6797	57
34	0.67450	85168	42426	63214	24609		1.48256	09685	12740	25478	7157	56
35	0.70020	75382	09709	77945	85227		1.42814	80067	42114	50216	0618	55
36	0.72654	25280	05360	88589	54668		1.37638	19204	71173	53820	7210	54
37	0.75355	40501	02794	15707	39564		1.32704	48216	20410	03715	9473	53
38	0.78128	56265	06717	39706	29500		1.27994	16321	93078	78031	1030	52
39	0.80978	40331	95007	14803	69914		1.23489	71565	35051	39855	6175	51
40	0.83909	96311	77280	01176	31273		1.19175	35925	94209	95870	5308	50
41	0.86928	67378	16226	66220	00956		1.15036	84072	21009	55587	6331	49
42	0.90040	40442	97839	94512	04772		1.11061	25148	29192	87014	3482	48
43	0.93251	50861	37661	70561	21856		1.07236	87100	24682	53294	6028	47
44	0.96568	87748	07074	04595	80273		1.03553	03137	90569	50695	8833	46
45	1.00000	00000	00000	00000	00000		1.00000	00000	00000	00000	0000	45
	cot $\theta$						tan $\theta$					$\theta^\circ$

Values of the two coefficients  $\sigma_4$  and  $\sigma_6$  and the three functions  $e_r$  are computed for  $\alpha = 10^\circ (10^\circ) 170^\circ$ . The results rounded off to 16D are shown in Tables 1 and 2. In these tables, (n) at the end of the number represents the factor  $10^n$ . It should also be pointed out that whenever the sign is ambiguous, the upper one corresponds to the upper value of  $\alpha$  and the lower with the lower value of  $\alpha$ . The computation was carried out, with adequate guard figures, on an IBM 1620 electronic computer, using a FORTRAN II program. Checking was performed by the method described in one of the previous papers [2]. It appears that when two values of  $\alpha$  are supplementary angles, the respective corresponding values of  $\sigma_4$ ,  $\sigma_6$ , and  $e_r$  are conjugate complex numbers.

Two coefficients deserve special mention. One is  $\sigma_4$  for  $\alpha = 90^\circ$ , which is equal to  $(4/15) [K(\sin 45^\circ)]^4$ ; the other is  $\sigma_6$  for  $\alpha = 60^\circ$  or  $120^\circ$ , which is equal to  $(64\sqrt{3}/315) [K(\sin 15^\circ)]^6$ , where  $K(k)$  represents the complete elliptic integral of the first kind in Legendre form, of modulus  $k$ . The values of these coefficients to 25D are:

$$(9) \quad \begin{aligned} \sigma_4 &= 3.15121\ 20021\ 53897\ 53821\ 76899, \\ \sigma_6 &= 5.86303\ 16934\ 25401\ 59797\ 02134. \end{aligned}$$

For the convenience of the reader, there is included in Table 3 a compilation of 25D values of the tangent and cotangent for arguments  $1^\circ(1^\circ)89^\circ$ . These data, which are required in computing the values of  $c$ , are here tabulated with the same range and precision as for the values of sine and cosine given by G. W. and R. M. Spenceley [4]. The only comparable table of decimal approximations to the tangent appears to be the relatively inaccessible 30D table of Herrmann [5].

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2. C. B. LING & C. P. TSAI, "Evaluation at half periods of Weierstrass' elliptic function with rhombic primitive period-parallelogram," *Math. Comp.*, v. 18, 1964, pp. 433-440. (Erratum: A negative sign should precede  $K_3$  in the first of Eqs. (7) on p. 434.)

3. C. B. LING, "Tables of values of  $\sigma_2$  relating to Weierstrass' elliptic function," *Math. Comp.*, v. 19, 1965, pp. 123-127.

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## The Asymptotic Expansion of the Integrals Psi and Chi In Terms of Tchebycheff Polynomials

By G. T. Thompson

Hummer [3] has given the expansion of the Dawson function which is used in the calculation of line-absorption coefficient due to Doppler effect and damping. Psi and Chi are two integrals used to determine the Doppler broadening effect on neutron cross sections in the resonance region [1]. They are functions of two variables and are given by

$$\begin{aligned} \psi(x, \theta) &= \frac{1}{\sqrt{4\theta}} \int_{-\infty}^{\infty} \frac{\exp [-(y-x)^2/4\theta]}{1+y^2} dy \\ \chi(x, \theta) &= \frac{1}{\sqrt{4\theta}} \int_{-\infty}^{\infty} \frac{y \exp [-(y-x)^2/4\theta]}{1+y^2} dy \end{aligned}$$

and satisfy these conditions

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