21, $\cdots$, 100 were computed by use of the "asymptotic" expansion given in (5). The "overlap" of the two methods for $n = 20, 21, \cdots, 25$ was provided as a numerical verification of this "asymptotic" expansion. It was found that these two methods, gave results which differed (in the eighth significant digit) by thirteen units for $n = 20$, four units for $n = 21$, and not more than one unit for $n = 22, 23, 24$, and 25.

Since the numerical integrations were computed in an ascending order, i.e., the integrand for $M_n^{(1)}$ was multiplied by t in order to obtain the integrand for $M_{n+1}^{(1)}$, $n = 2, 3, \cdots, 24$, and, in view of the agreement indicated above, it is felt that all values in Table 2 are correct, except for possible rounding errors of one unit in the eighth significant digit.

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Bessel-Function Identities Needed for the Theory of Axisymmetric Gravity Waves

By Lawrence R. Mack

1. Introduction. Certain identities involving integrals of products of Bessel functions are required for analyses of finite-amplitude axisymmetric gravity waves [3], [4]. The specific identities needed through the third-order wave solution are of two distinct types. The first type equates to zero the sum of two or three integrals of products of several Bessel functions, all integrands in a particular identity being products of the same number of Bessel functions. Of the required identities of this type the one with products of two Bessel functions is trivial, while those whose integrands are products of three Bessel functions are obtainable from the results of Fettis [1]. Each identity of the second type equates an integral of the product of four Bessel functions to the sum of an infinite number of products of pairs of

Received November 23, 1964.

integrals in which each integrand is the product of three Bessel functions. Two general identities are derived here, from which the special cases needed through the third-order gravity-wave theory are obtained. The first general identity should also yield all identities of the first type which will be needed for higher-order wave solutions. Both the general identities and the special cases may also be applicable to other problems in which expansions in the Dini type of Fourier-Bessel series are used.

2. First General Identity. Let $y(r)$ be the nonsingular solution $J_0(Kr)$ of the Bessel equation of order zero

$$y'' + \frac{1}{r} y' + K^2 y = 0, \qquad r \geqq 0, \tag{1}$$

where primes denote differentiation with respect to r. Let $u(r)$ be any function with continuous derivative in the range $0 \leqq r \leqq 1$. Multiplication of (1) by ru and integration by parts from 0 to 1 yields the general identity of the first type

$$\int_0^1 r[u'y' - K^2 uy]\, dr = u(1)y'(1). \tag{2}$$

The K in equation (2) need not be an eigenvalue.

3. Dini Series. Let K_n, $n \geqq 0$, be the eigenvalues for which

$$J_1(K_n) = 0, \qquad K_n \geqq 0, \tag{3}$$

arranged in ascending order of magnitude beginning with $K_0 = 0$. Let us introduce the shortened notation

$$J_{mn} \equiv J_m(K_n r).$$

In terms of the eigenvalues given by (3), the Dini expansion of an arbitrary function $F(r)$ [5, Chapter 18] is

$$F(r) = \sum_{n=0}^{\infty} \alpha_n(F) J_{0n} \tag{4}$$

with the coefficients $\alpha_n(F)$ given by

$$\alpha_n(F) = \frac{\int_0^1 rF(r)J_{0n}\, dr}{\frac{1}{2}J_0^2(K_n)}, \qquad n \geqq 0 \tag{5}$$

where use has been made of the well-known orthogonality relation

$$\int_0^1 rJ_{0n}\, J_{0s}\, dr = \begin{cases} \frac{1}{2}J_0^2(K_s) > 0, & n = s, \\ 0, & n \neq s. \end{cases} \tag{6}$$

It has been shown by Watson [5, Arts. 18.33, 18.35, and 18.55] that, if $F(r)$ is continuous and has limited total fluctuation in the interval $0 \leqq r \leqq 1$, the Dini expansion of $F(r)$, equations (4) and (5), will converge uniformly to $F(r)$ in that interval.

4. Second General Identity. If $G(r)$ and $H(r)$ are continuous and have limited total fluctuation in the interval $0 \leqq r \leqq 1$, each has a uniformly convergent Dini expansion in that interval. Their product will likewise have a uniformly convergent Dini expansion

$$G(r)H(r) = \sum_{n=}^{\infty} \alpha_n(GH)J_{0n}\,. \tag{7}$$

Substitution of the expansion for $G(r)$ and $H(r)$ into the left side of (7) and multiplication of the two series expansions, gives

$$\sum_{n=0}^{\infty} \alpha_n(GH)J_{0n} = \sum_{m=0}^{\infty}\left[\alpha_m(G)\sum_{p=0}^{\infty}\alpha_p(H)J_{0m}\,J_{0p}\right] \tag{8}$$

as a possible arrangement of terms. Multiplication of (8) by rJ_{0s}, integration from 0 to 1, and use of both (6) and the relation

$$\int_0^1 rJ_{0m}\,J_{0p}\,J_{0s}\,dr = \tfrac{1}{2}J_0^{\,2}(K_s)\alpha_s(J_{0m}\,J_{0p})$$

yields the second general identity

$$\alpha_n(GH) = \sum_{m=0}^{\infty}\left[\alpha_m(G)\sum_{p=0}^{\infty}\alpha_p(H)\alpha_n(J_{0m}\,J_{0p})\right], \qquad n \geqq 0. \tag{9}$$

We have established (9) by formal procedures. To prove that the double infinite series, summed in the indicated order, converges to $\alpha_n(GH)$, we may use the generalized Parseval's theorem [2, p. 761] to show that

$$\sum_{p=0}^{\infty} \alpha_p(H)\alpha_n(J_{0m}J_{0p}) \quad \text{converges to } \alpha_n(HJ_{0m})$$

and that

$$\sum_{m=0}^{\infty} \alpha_m(G)\alpha_n(HJ_{0m}) \quad \text{converges to } \alpha_n(GH)$$

each for $n \geqq 0$.

5. Special Cases for Gravity-Wave Theory. Let K in the first general identity (2) be restricted to the positive eigenvalues K_n, $n \geqq 1$, which satisfy (3). Let $u = vwx$; the special cases of (2) needed for the theory of axisymmetric gravity waves [3], [4] involve Lommel-type integrals and are obtained by choosing $v(r)$, $w(r)$, and $x(r)$ from among the functions J_{0m}, J_{1m}, and J_{00} (i.e., unity). If the operator $I(F)$ is defined as

$$I(F) \equiv \int_0^1 rF(r)\,dr$$

these identities become

$$I(J_{01}^4) - 3I(J_{01}^2\,J_{11}^2) = 0, \tag{10}$$

$$3I(J_{01}^2\,J_{11}^2) - 2I\left(J_{01}\,\frac{J_{11}^3}{K_1\,r}\right) - I(J_{11}^4) = 0, \tag{11}$$

$$K_1 I(J_{01}^2 J_{0n}) - K_1 I(J_{11}^2 J_{0n}) - K_n I(J_{01} J_{11} J_{1n}) = 0, \tag{12}$$

$$K_n I(J_{01}^2 J_{0n}) - 2K_1 I(J_{01} J_{11} J_{1n}) = 0, \tag{13}$$

$$3I(J_{01} J_{11}^2) - 2I\left(\frac{J_{11}^3}{R_1 r}\right) = 0, \tag{14}$$

$$I(J_{0n}^2) - I(J_{1n}^2) = 0. \tag{15}$$

Each integral of (15) may be individually evaluated as $\frac{1}{2}J_0^2(K_n)$. The α and I notations are not independent but are related by

$$\alpha_n(F) = \frac{I(FJ_{0n})}{\frac{1}{2}J_0^2(K_n)} = \frac{I(FJ_{0n})}{I(J_{0n}^2)}.$$

Let $n = 0$ and let $G(r)$, $H(r)$ be chosen from the functions J_{11}^2, J_{01}^2; then the second general identity (9) yields

$$I(J_{11}^4) = 2\sum_{m=0}^{\infty} \frac{[I(J_{11}^2 J_{0m})]^2}{J_0^2(K_m)}, \tag{16}$$

$$I(J_{01}^2 J_{11}^2) = 2\sum_{m=0}^{\infty} \frac{I(J_{01}^2 J_{0m}) I(J_{11}^2 J_{0m})}{J_0^2(K_m)}, \tag{17}$$

$$I(J_{01}^4) = 2\sum_{m=0}^{\infty} \frac{[I(J_{01}^2 J_{0m})]^2}{J_0^2(K_m)}. \tag{18}$$

Two linear combinations of (16), (17), (18), making use also of some of the identities of the first type, are the identities of the second type required for the third-order wave solution. Numerical computations show that in both of these combinations the sum of the first three terms is over 99.9% of the total.

An additional special case of (9) which may be useful for numerical computations is obtained by letting $H(r) = J_{0j}$ and noting that $\alpha_p(J_{0j})$ equals δ_{pj}, the Kronecker delta. Thus

$$\alpha_n(GJ_{0j}) = \sum_{m=0}^{\infty} \alpha_m(G)\alpha_n(J_{pm}J_{0j}). \tag{19}$$

The author wishes to thank Dr. T. W. Ting for his valuable suggestions and to acknowledge the financial support, through a released-time grant, of The University of Texas Research Institute.

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