

# Computation of the Riemann Function for the Operator $\partial^n/\partial x_1 \partial x_2 \cdots \partial x_n + a(x_1, x_2, \dots, x_n)$

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**1. Introduction.** Riemann's method [1] is, in principle, well suited for the numerical solution of boundary value problems for the hyperbolic differential equation

$$(1.1) \quad \partial^2 u / \partial x_1 \partial x_2 + a(x_1, x_2)u = F(x_1, x_2).$$

One first finds the Riemann function, which is the solution of a homogeneous adjoint equation subject to simple boundary conditions that are independent of the given boundary data. The solution of (1.1), for any appropriate boundary data, can then be obtained by evaluating a definite integral, where the Riemann function and the boundary data appear in the integrand.

The main obstacle to the use of this method for numerical computation has been the difficulty in finding the Riemann function. It was pointed out in a 1947 paper by Cohn [2] that, aside from  $a(x_1, x_2) = \text{constant}$  and  $a(x_1, x_2) = k(k - 1) \cdot (x_1 + x_2)^{-2}$ , which was treated by Riemann, there are very few cases where expressions for the Riemann function have been obtained. One can, of course, use the Picard method of successive approximations, but this is usually not practical. In this paper we derive a simple recurrence formula for the Riemann function, for the case where  $a(x_1, x_2)$  can be expanded in a Taylor series about the point where the solution is sought.

We consider here the Riemann function for the operator  $L$  in the  $n$  dimensional analogue of (1.1),

$$(1.2) \quad L(u) = \partial^n u / \partial x_1 \partial x_2 \cdots \partial x_n + a(x_1, x_2, \dots, x_n)u = F(x_1, x_2, \dots, x_n).$$

It was shown by Bianchi [3], [4], [5] and Niccoletti [6], who treated the more general linear hyperbolic equation containing pure mixed derivatives of all orders through the  $n$ th, that Riemann's method can be extended to this case. The solutions of the characteristic value and the Cauchy problem for (1.2) are discussed in detail in [7]. The Riemann function  $v$  for the operator  $L$  in (1.2) can be defined by the integral equation

$$(1.3) \quad \begin{aligned} v(x_1, x_2, \dots, x_n : \beta_1, \beta_2, \dots, \beta_n) \\ = 1 - \int_{x_1}^{\beta_1} \int_{x_2}^{\beta_2} \cdots \int_{x_n}^{\beta_n} a(\xi_1, \xi_2, \dots, \xi_n) \\ \cdot v(\xi_1, \xi_2, \dots, \xi_n : \beta_1, \beta_2, \dots, \beta_n) d\xi_1 d\xi_2 \cdots d\xi_n. \end{aligned}$$

The solution of the characteristic value problem for (1.2) can, for example, be written

$$(1.4) \quad \begin{aligned} u(\beta_1, \beta_2, \dots, \beta_n) = \bar{u}(\beta_1, \beta_2, \dots, \beta_n) \\ + \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \cdots \int_{\alpha_n}^{\beta_n} v(x_1, x_2, \dots, x_n : \beta_1, \beta_2, \dots, \beta_n) \\ \cdot [F - a\bar{u}](x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n, \end{aligned}$$

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where  $\bar{u}$  is the solution of  $\partial^n u / \partial x_1 \partial x_2 \cdots \partial x_n = 0$  satisfying the data given on the hyperplanes  $x_i = \alpha_i$ .

The present discussion is confined to the case where the function  $a(x_1, x_2, \dots, x_n)$  is expandable about the point  $(\beta_1, \beta_2, \dots, \beta_n)$  in a Taylor series valid in the region  $|x_i - \beta_i| < r_i'$ . This is equivalent to the assumption that  $a(x_1, x_2, \dots, x_n)$  is an analytic function of  $n$  complex variables. The Taylor series for  $v$  then has the same region of convergence as the series for  $a$  (Section 4). Less stringent regularity conditions can be applied to the function  $F$  and the given boundary data.

**2. The Recurrence Formula.** To fix the ideas, the recurrence formula for  $v$  will be obtained for the one dimensional case first. Suppose  $v$  is expandable in the Taylor series

$$(2.1) \quad v = 1 + \sum_{i=1}^{\infty} f_i (x - \beta)^i, \quad f_i = \frac{1}{i!} \left. \frac{\partial^i v}{\partial x^i} \right]_{x=\beta}.$$

From (1.3),

$$(2.2) \quad \begin{aligned} \partial v / \partial x ]_{x=\beta} &= [a(x)v(x;\beta)]_{x=\beta} = a(\beta), \\ \partial^2 v / \partial x^2 ]_{x=\beta} &= \partial(av) / \partial x ]_{x=\beta} = [a\partial v / \partial x + v\partial a / \partial x]_{x=\beta}, \\ \partial^i v / \partial x^i ]_{x=\beta} &= \partial^{i-1}(av) / \partial x^{i-1} ]_{x=\beta} = a(\beta)\partial^{i-1}v / \partial x^{i-1} ]_{x=\beta} + \dots \end{aligned}$$

This method for finding the coefficient  $f_i$  in (2.1), from the previous ones, was proposed by du Bois Reymond [8] for the two dimensional case.

Since all the derivatives  $\partial^i a / \partial x^i$  are required in (2.2), one might as well use the essentially identical, but simpler, procedure which consists of taking

$$(2.3) \quad a(x) = \sum_{i=0}^{\infty} a_i (x - \beta)^i, \quad a_i = \frac{1}{i!} \left. \frac{\partial^i a}{\partial x^i} \right]_{x=\beta},$$

together with (2.1), inserting these in (1.3) and equating coefficients. The result is the recurrence formula

$$(2.4) \quad if_i = \sum_{k=0}^{i-1} a_{i-1-k} f_k.$$

For the higher dimensional cases, the analogue of (2.4) is a convenient and practical way to get the Riemann function. Its merit lies in the fact that the same coefficients are not computed over and over again, as in the method of successive approximations.

For the Taylor series expansion of  $v$  in the  $n$  dimensional case let

$$(2.5) \quad b(x_1, x_2, \dots, x_n) = (-1)^{n+1} a(x_1, x_2, \dots, x_n),$$

so that, from (1.3)

$$(2.6) \quad v_{,12\dots n} = bv, \quad v(x_1, x_2, \dots, x_{i-1}, \beta_i, x_{i+1}, \dots, x_n; \beta_1, \beta_2, \dots, \beta_n) = 1.$$

Let

$$(2.7) \quad b = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} b_{i_1 i_2 \dots i_n} (x_1 - \beta_1)^{i_1} (x_2 - \beta_2)^{i_2} \cdots (x_n - \beta_n)^{i_n},$$

and

$$(2.8) \quad v = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f_{i_1 i_2 \dots i_n} (x_1 - \beta_1)^{i_1} (x_2 - \beta_2)^{i_2} \cdots (x_n - \beta_n)^{i_n},$$

where

$$(2.9) \quad b_{i_1 i_2 \dots i_n} = \frac{1}{i_1! i_2! \cdots i_n!} \frac{\partial^{i_1+i_2+\dots+i_n} b}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \Big|_{x_k=\beta_k, k=1, 2, \dots, n},$$

and

$$(2.10) \quad f_{i_1 i_2 \dots i_n} = \frac{1}{i_1! i_2! \cdots i_n!} \frac{\partial^{i_1+i_2+\dots+i_n} v}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \Big|_{x_k=\beta_k, k=1, 2, \dots, n}.$$

Here, from (1.3),  $f_{0\dots 0} = 1$ , and any other  $f_{i_1 i_2 \dots i_n}$  vanishes if at least one of the  $i$ 's is zero.

The multiplication of (2.7) and (2.8) and the insertion of the result in (2.6) leads to the recurrence formula for the coefficients in (2.8), namely,

$$(2.11) \quad i_1 i_2 \cdots i_n f_{i_1 i_2 \dots i_n} = \sum_{k_1=0}^{i_1-1} \sum_{k_2=0}^{i_2-1} \cdots \sum_{k_n=0}^{i_n-1} b_{i_1-1-k_1, i_2-1-k_2, \dots, i_n-1-k_n} f_{k_1 k_2 \dots k_n}.$$

Once the Riemann function, corresponding to a particular function  $a(x_1, x_2, \dots, x_n)$ , is known, the characteristic boundary value problem, for example, is reduced to one of evaluating a multiple integral [see (1.4)]. The numerical computation proceeds as follows: For a fixed point  $(\beta_1, \beta_2, \dots, \beta_n)$  at which the solution is sought, the coefficients  $f_{i_1 i_2 \dots i_n}$  are calculated, to any desired accuracy, from the recurrence formula (2.11). In general, these coefficients will be functions of  $\beta_1, \beta_2, \dots, \beta_n$ . The grid points  $(x_1, x_2, \dots, x_n)$  to be used in evaluating the integral in (1.4) are then selected. For each of these points  $v(x_1, x_2, \dots, x_n; \beta_1, \beta_2, \dots, \beta_n)$  is calculated, using (2.8), and stored. The solution of (1.2) at the point  $(\beta_1, \beta_2, \dots, \beta_n)$  is then obtained for any appropriate set of boundary data, by the numerical integration of (1.4), using the stored values of  $v$ . The same  $v$  can also be used for the solution of the Cauchy problem for (1.2). Here, too, the solution can be expressed in terms of an integral containing  $v$  [7].

**3. Examples.**

A. To test the method, the recurrence formula (2.11) is used here to calculate the Riemann function  $v$  for a case where  $v$  is known explicitly, namely,

$$(3.1) \quad L(u) = \partial^2 u / \partial x_1 \partial x_2 - (2/(x_1 + x_2)^2)u.$$

Here, the Riemann function, which must satisfy

$$(3.2) \quad \partial^2 v / \partial x_1 \partial x_2 - [2/(x_1 + x_2)^2]v = 0, \quad v(x_1, x_2; \beta_1, \beta_2) = v(\beta_1, x_2; \beta_1, \beta_2) = 1.$$

is

$$(3.3) \quad v(x_1, x_2; \beta_1, \beta_2) = 2 \frac{(x_2 + \beta_1)(x_1 + \beta_2)}{(x_1 + x_2)(\beta_1 + \beta_2)} - 1.$$

For comparison with (3.3), (2.11) and (2.7) were used to calculate  $v(x_1, x_2; 1, 1)$

TABLE I  
 The coefficients  $f_{i,j}$  in the expansion of the Riemann function  $v(x_1, x_2; 1, 1)$  for the operator  $\partial^2/\partial x_1 \partial x_2 - 2/(x_1 + x_2)^2$

$\frac{i_1}{i_2}$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	50000000	25000000	12500000	6250000	3125000	1562500	781250	390625	195312	97656
2	0	25000000	25000000	18750000	12500000	7812500	4687500	2734375	1562500	8906	488281
3	0	12500000	18750000	18750000	15625000	11718750	8203125	5468750	3515625	2197266	1342773
4	0	6250000	12500000	15625000	15625000	13671875	10937500	8203125	5859375	4028320	2685547
5	0	3125000	7812500	11718750	13671875	13671875	12304687	10253906	8056640	6042480	4364014
6	0	1562500	4687500	8203125	10937500	12304687	12304787	11279297	9667969	7855224	6109619
7	0	781245	2734375	5468750	8203125	10253906	11279297	11279297	10473633	9164429	7637024
8	0	390625	1562500	3515625	5859375	8056640	9667969	10473633	10473633	9819031	8728027
9	0	195312	8906	2197266	4028320	6042480	7855224	99165529	9819031	9819031	9273529
10	0	97656	488281	1342773	2685547	4364014	6109619	7637024	8728027	9273529	9273529

TABLE II

The  $n$ th partial sums and the exact values of the Riemann function  $v(x_1, x_2 : 1, 1)$  for the operator  $\partial^2/\partial x_1 \partial x_2 - 2/(x_1 + x_2)^2$ .

	$(x_1, x_2)$			
	$(.8, .8)$	$(.6, .6)$	$(.4, .4)$	$(.2, .2)$
$n = 1$	1.020000	1.080000	1.180000	1.320000
2	1.024400	1.118400	1.320400	1.678400
3	1.024932	1.129408	1.390708	1.952832
4	1.024992	1.132326	1.423464	2.151078
5	1.024999	1.133078	1.438273	2.290932
6	1.025000	1.133269	1.444859	2.388318
7	1.025000	1.133317	1.447759	2.455571
8	1.025000	1.133329	1.449027	2.501749
9	1.025000	1.133332	1.449579	2.533320
10	1.025000	1.133333	1.449818	2.554834
Exact value	1.025000	1.133333	1.450000	2.600000

for several points  $(x_1, x_2)$ . In two dimensions, the recurrence formula (2.11) is

$$(3.4) \quad i_1 i_2 f_{i_1 i_2} = \sum_{k_1=0}^{i_1-1} \sum_{k_2=0}^{i_2-1} b_{i_1-1-k_1, i_2-1-k_2} f_{k_1 k_2},$$

$$f_{00} = 1, \quad f_{0j} = f_{j0} = 0 \quad \text{for } j \neq 0.$$

For  $b = 2/(x_1 + x_2)^2$  the coefficients in (2.7) are

$$(3.5) \quad b_{ij} = 2(-1)^{i+j}(i+j+1)!/[i!j!(\beta_1 + \beta_2)^{2+i+j}].$$

The coefficients  $f_{i_1 i_2}$  in the expansion (2.8) for  $v(x_1, x_2; 1, 1)$  are listed for  $i_1, i_2 \leq 10$  in Table I. In Table II, the exact values of  $v(x_1, x_2; 1, 1)$ , obtained with (3.3), are listed for several values of  $(x_1, x_2)$ , together with the  $n$ th partial sums calculated from

$$(3.6) \quad v^{(n)} = 1 + \sum_{i_1=1}^n \sum_{i_2=1}^n f_{i_1 i_2} (x_1 - 1)^{i_1} (x_2 - 1)^{i_2}.$$

B. It is evident from (1.4), and the analogous formula for the solution of the Cauchy problem [7], that the value of the solution of (1.2) at the single point  $(\beta_1, \beta_2, \dots, \beta_n)$  can be found, for any appropriate boundary data, if the Riemann function  $v$  is known for that point. Consider the operator

$$(3.7) \quad \partial^n/\partial x_1 \partial x_2 \cdots \partial x_n + a(\sigma),$$

where

$$(3.8) \quad \sigma = x_1 x_2 \cdots x_n.$$

To find the Riemann function  $v$  for  $(\beta_1, \beta_2, \dots, \beta_n) = (0, 0, \dots, 0)$  let

$$(3.9) \quad b(\sigma) = (-1)^{n+1} a(\sigma),$$

$$(3.10) \quad b = \sum_{i=0}^{\infty} b_i \sigma^i,$$

and

$$(3.11) \quad v = 1 + \sum_{i=1}^{\infty} f_i \sigma^i.$$

For this special case, the recurrence formula (2.11) becomes

$$(3.12) \quad i^n f_i = \sum_{k=0}^{i-1} b_{i-1-k} f_k.$$

Now suppose

$$(3.13) \quad b = \frac{K_1}{1 - K_2 \sigma},$$

where  $K_1$  and  $K_2$  are constants. Here, the coefficients in (3.10) are

$$(3.14) \quad b_i = K_1 K_2^i, \quad |K_2 \sigma| < 1.$$

From (3.14) and the recurrence formula (3.12),

$$(3.15) \quad i^n f_i = [K_1 + K_2(i - 1)^n] f_{i-1}.$$

Now, from (3.15),

$$(3.16) \quad \begin{aligned} (i - 1)^n f_{i-1} &= [K_1 + K_2(i - 2)^n] f_{i-2}, \\ (i - 2)^n f_{i-2} &= [K_1 + K_2(i - 3)^n] f_{i-3}, \\ &\dots\dots\dots \\ 2f_2 &= (K_1 + K_2)f_1, \\ f_1 &= K_1. \end{aligned}$$

Multiplication of the terms in (3.15) and (3.16) leads to the coefficients in the expansion for  $v(x_1, x_2, \dots, x_n; 0, 0, \dots, 0)$ , namely,

$$(3.17) \quad f_i = K_1(K_1 + K_2)(K_1 + 2^n K_2) \dots (K_1 + (i - 1)^n K_2) / [i!]^n.$$

The substitution  $v = v(\sigma)$  in (2.6), with  $a = (-1)^{n+1} K_1 / (1 - K_2 \sigma)$ , results in an ordinary differential equation for which

$$(3.18) \quad v = 1 + \sum_{i=1}^{\infty} f_i x^i,$$

with  $f_i$  given by (3.17), is a particular solution. For example, in two dimensions (3.18) is the solution of the hypergeometric equation

$$(3.19) \quad \sigma \frac{d^2 v}{d\sigma^2} + \frac{dv}{d\sigma} - \frac{K_1}{1 - K_2 \sigma} v = 0, \quad v(0) = 1,$$

and for  $n = 3$ , (3.18) is the solution of

$$(3.20) \quad \sigma^2 \frac{d^3 v}{d\sigma^3} + 3\sigma \frac{d^2 v}{d\sigma^2} + \frac{dv}{d\sigma} + \frac{K_1}{1 - K_2 \sigma} v = 0, \quad v(0) = 1.$$

**4. The Convergence of the Expansion for the Riemann Function.** The series (2.8) for the Riemann function will converge in the open region about

$(\beta_1, \beta_2, \dots, \beta_n)$  where (2.7), the expansion for  $b$ , converges. This can be shown as follows: Suppose that (2.7) converges in the region  $|x_i - \beta_i| \leq r_i'$ . Let  $M$  be the maximum  $|b_{i_1 i_2 \dots i_n}|$  in this region and let

$$(4.1) \quad B = \frac{M}{\left[1 - \frac{x_1 - \beta_1}{r_1}\right] \left[1 - \frac{x_2 - \beta_2}{r_2}\right] \dots \left[1 - \frac{x_n - \beta_n}{r_n}\right]},$$

where  $0 < r_i < r_i'$ . Denote the expansion for  $B$  in  $|x_i - \beta_i| < r_i$  by

$$(4.2) \quad B = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \dots \sum_{i_n=0}^{\infty} B_{i_1 i_2 \dots i_n} (x_1 - \beta_1)^{i_1} (x_2 - \beta_2)^{i_2} \dots (x_n - \beta_n)^{i_n},$$

and confine the discussion to the region  $|x_i' - \beta_i| < r_i < r_i'$ . Here (4.2) dominates (2.7), i.e.,

$$(4.3) \quad B_{i_1 i_2 \dots i_n} \geq |b_{i_1 i_2 \dots i_n}|.$$

The solution of

$$(4.4) \quad \partial^n V / \partial x_1 \partial x_2 \dots \partial x_n = BV, \\ V(x_1, x_2, \dots, x_{i-1}, \beta_i, x_{i+1}, \dots, x_n; \beta_1, \beta_2, \dots, \beta_n) = 1$$

can, in its domain of convergence, be written

$$(4.5) \quad V = 1 + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \dots \sum_{i_n=1}^{\infty} F_{i_1 i_2 \dots i_n} (x_1 - \beta_1)^{i_1} (x_2 - \beta_2)^{i_2} \dots (x_n - \beta_n)^{i_n}.$$

The coefficients in (4.5) are obtained from (2.11) as

$$(4.6) \quad i_1 i_2 \dots i_n F_{i_1 i_2 \dots i_n} = \sum_{k_1=0}^{i_1-1} \sum_{k_2=0}^{i_2-1} \dots \sum_{k_n=0}^{i_n-1} B_{i_1-1-k_1, i_2-1-k_2, \dots, i_n-1-k_n} F_{k_1 k_2 \dots k_n}.$$

Since the coefficients  $B_{i_1 i_2 \dots i_n}$  are positive, it follows from (4.3) and (4.6) that (4.5) dominates (2.8), i.e.,

$$(4.7) \quad F_{i_1 i_2 \dots i_n} \geq |f_{i_1 i_2 \dots i_n}|.$$

To find the region of convergence of (4.5), consider the constant coefficient equation

$$(4.8) \quad \partial^n V / \partial \xi_1 \partial \xi_2 \dots \partial \xi_n = MV.$$

Bianchi [3] obtained the solution of (4.8) as

$$(4.9) \quad V(\xi_1, \xi_2, \dots, \xi_n; \beta_1, \beta_2, \dots, \beta_n) \\ = 1 + \sum_{j=1}^{\infty} [M^j [(\xi_1 - \beta_1)(\xi_2 - \beta_2) \dots (\xi_n - \beta_n)]^j / (j!)^n].$$

This result, (4.9), is also evident as a special case of Example B, Section 3, with  $K_2 = 0$ . The transformations

$$(4.10) \quad \xi_i - \beta_i = -r_i \ln \left[1 - \frac{x_i - \beta_i}{r_i}\right] = \sum_{k=1}^{\infty} \frac{(x_i - \beta_i)^k}{k r_i^{k-1}}$$

convert (4.8) into (4.4). From (4.9) and (4.10), the solution of (4.4) can be written

$$(4.11) \quad V = 1 + \sum_{j=1}^{\infty} \left\{ (1/j!)^n \left[ (-1)^n M r_1 r_2 \cdots r_n \prod_{i=1}^n \ln \left( 1 - \frac{x_i - \beta_i}{r_i} \right) \right]^j \right\}.$$

Since (4.5) is equivalent to (4.11), it converges in the region  $|x_i - \beta_i| < r_i$ . The series (2.8) for the Riemann function, since it is dominated by (4.5), must therefore converge in the region  $|x_i - \beta_i| < r_i < r_i'$ .

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