

A Class of Expansions of G -functions and the Laplace Transform

By Arun Verma

1. Introduction. Several general expansions involving G -functions were given in a series of papers by Meijer [4]. Recently, Wimp and Luke [6] obtained some expansions involving G -functions which were more general than the results of Meijer. These results were obtained by generalising certain known results of Field and Wimp [3]. The object of this paper is to use the Laplace transform and its inverse to derive certain types of expansions involving G -functions. The advantage of this method, in the present context, is to show how the expansions involving G -functions follow as obvious and natural consequences of similar expansions for the generalised hypergeometric functions.*

In §3 an expansion involving G -function has been proved by using a generalisation of a result due to Niblett [5] deduced in §2. The expansion is incidentally a generalisation of some of the expansions given by Wimp and Luke,† and contains as special cases many of the expansion theorems of Meijer [4].

2. In the first instance we shall prove the following generalisation of a result due to Niblett [5].

$$(1) \quad {}_{p+s}F_q \left[\begin{matrix} (a_p), (b_s); \\ (c_q) \end{matrix}; xw \right] = h \sum_{n=0}^{\infty} \frac{[h - n\alpha + 1]_{n-1} [(b_s)]_n [(e_u)]_n (-x)^n}{n! [(c_q)]_n} \\ \times {}_{p+2}F_{u+2} \left[\begin{matrix} (a_p), 1 + h(1 - \alpha)^{-1}, -n; \\ (e_u), h(1 - \alpha)^{-1}, h - n\alpha + 1 \end{matrix}; w \right] \\ \times {}_{s+u+1}F_q \left[\begin{matrix} (b_s) + n, (e_u) + n, h + n(1 - \alpha); \\ (c_q) + n \end{matrix}; x \right],$$

provided $p + s \leq q$, or $p + s = 1 + q$, and $|xw| < 1$, $s + u + 1 \leq q$, or $s + u = q$ and $|x| < 1$, and the series of the hypergeometric functions on the right hand is absolutely convergent. To prove (2.1) we use a simple extension of the method used earlier by Chaundy [1] for proving a similar result.

Comparing the coefficient of $[(a_p)]_N w^N / N!$ on both the sides of (2.1), we get

$$\frac{[(b_s)]_N x^N}{[(c_q)]_N} = \{h + N(1 - \alpha)\} \sum_{n=N}^{\infty} \frac{[h - n\alpha + 1]_{n-1} [(b_s)]_n [(e_u)]_n (-x)^n}{n! [(c_q)]_n} \\ \times \frac{[-n]_N}{[h - n\alpha + 1]_N} {}_{s+u+1}F_q \left[\begin{matrix} (b_s) + n, (e_u) + n, h + n(1 - \alpha); \\ (c_q) + n \end{matrix}; x \right].$$

Writing $n = N + r$, we find that this reduces to

$$1 = \{h + N(1 - \alpha)\} \sum_{r=0}^{\infty} \frac{[h + N(1 - \alpha) + 1 - \alpha]_{r-1} [(b_s) + N]_r [(e_u) + N]_r (-x)^r}{r! [(c_q) + N]_r} \\ \times {}_{s+u+1}F_q \left[\begin{matrix} (b_s) + N + r, (e_u) + N + r, h + (1 - \alpha)(N + r); \\ (c_q) + N + r \end{matrix}; x \right].$$

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* For the notation and the properties of G -functions see Erdélyi [2].

† I am grateful to Mr. Y. L. Luke and the referee for suggesting certain improvements in the original version of this paper.

The term independent of x on the right hand side is seen to be unity. It thus remains to show that the coefficient of any positive power of x vanishes on the right, i.e., when $M > 0$,

$$\frac{[(b_s) + N]_M [(e_u) + N]_M}{[(c_q) + N]_M} \sum_{r=0}^M \frac{(-)^r [h + N(1 - \alpha) + 1 - r\alpha]_{M-1}}{r! [M - r]!} = 0.$$

This, however, is the coefficient of x^{M-1} in

$$\frac{[(b_s) + N]_M [(e_u) + N]_M}{M! [(c_q) + N]_M} (1 - x)^{-h - N(1 - \alpha) - 1} [1 - (1 - x)^\alpha]^M,$$

in which the lowest term is x^M . This completes the formal proof of (2.1). The rearrangement of the infinite series is justified due to absolute convergence.

This result reduces to one proved by Niblett [5] for $w = 1$ and $u = q$. And for $\alpha = 0$, this gives us a result due to Meijer [4] on assuming $u = q$, $(e_u) = (c_q)$ and after suitable adjustments of the parameters.

3. Next, we generalise (2.1) for the G -functions in the following form:

$$\begin{aligned} G_{1+l+q_1+W, h+s+m}^{h+s+t, 1+v} \left(xy \left| \begin{matrix} 1, (k_l), (c_q), (\delta_w) \\ (a_p), (b_s), (g_m) \end{matrix} \right. \right) \\ = h\Gamma[(a_p); (e_u), (\delta_w)] \sum_{n=0}^{\infty} \frac{1}{n! \Gamma[1 - \alpha n + h]} \\ \times {}_{p+2}F_{w+u+2} \left[\begin{matrix} (a_p), 1 + h(1 - \alpha)^{-1}, -n; 1/w \\ (\delta_w), (e_u), h(1 - \alpha)^{-1}, h - \alpha n + 1 \end{matrix} \right] \\ \times G_{1+l+q, 1+s+u+m}^{1+s+u+t, 1+v} \left(x \left| \begin{matrix} 1 - n, (k_l), (c_q) \\ (b_s), (e_u), h - \alpha n, (g_m) \end{matrix} \right. \right), \end{aligned} \tag{1}$$

provided

$$v \leq l, t \leq m, l + m + q + W < 1 + h + s + 2t + 2v, |\arg xw| <$$

$$\frac{1}{2}\pi(1 + h + s + 2t + 2v - l - m - q - W), t + v < 2 + 2l +$$

$$2m + 2q + s + u, |\arg x| < \frac{1}{2}\pi(2 + 2l + 2m + 2q + s + u - t - v),$$

and the series on the right hand side has a meaning.

We prove (3.1) by mathematical induction. We suppose that (3.1) is true for some fixed values of l, m, p, q, s, t, u, v and W . To effect the induction with respect to v multiply both sides by z^{-f_v+1} , replace x by xz and take the Laplace transform with respect to z on both the sides. Then using the known result

$$\int_0^\infty e^{-y} y^{-\alpha} G_{p,q}^{m,n} \left(xy \left| \begin{matrix} (a_p) \\ (b_q) \end{matrix} \right. \right) dy = G_{p+1,q}^{m,n+1} \left(x \left| \begin{matrix} \alpha, (a_p) \\ (b_q) \end{matrix} \right. \right), \tag{2}$$

provided

$$(i) p + q < 2(m + n), |\arg x| < \pi(m + n - \frac{1}{2}p - \frac{1}{2}q),$$

$$(ii) Rl\alpha < Rlb_h + 1, h = 1, 2, \dots, m;$$

on both the sides, we get a relation in which v has been replaced by $v + 1$.

Further, to effect the induction with respect to m multiply both sides by $z^{\theta_{m+1}-1}$,

replace x by x/z and take the inverse Laplace transform with respect to z on both the sides. Then using the known result

$$(3) \quad \frac{1}{2\pi i} \int_c e^{t\alpha-1} G_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_p) \\ (b_q) \end{matrix} \right. \right) dt = G_{p,q+1}^{m,n} \left(x \left| \begin{matrix} (a_p) \\ (b_q), \alpha \end{matrix} \right. \right),$$

provided $Re\alpha > 0$, and the conditions (3.2-i) and (3.2-ii) hold; on both the sides, we get a relation in which m has been replaced by $m + 1$.

Similarly, the induction with respect to l, N, p, q, s, t, u and W can be effected. Since, for $l = m = t = v = W = 0$, (3.1) reduces to the relation (2.1), the result is established completely.*

Next, using first the relation [2; 5.3.1 (9)]

$$G_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_p) \\ (b_q) \end{matrix} \right. \right) = G_{q,p}^{n,m} \left(x^{-1} \left| \begin{matrix} 1 - (b_q) \\ 1 - (a_p) \end{matrix} \right. \right),$$

and then

$$G_{p+1,q+1}^{m+1,k} \left(x \left| \begin{matrix} (a_p), 1 - n \\ 1, (b_q) \end{matrix} \right. \right) = (-)^n G_{p+1,q+1}^{m,k+1} \left(x \left| \begin{matrix} 1 - n, (a_p) \\ (b_q), 1 \end{matrix} \right. \right),$$

which is apparent from the definition of the G -function, on both the sides of (3.1) one can get an expansion of $G(\lambda x)$ in a series of the products of $G(x)$ and $F(\lambda)$.

It is worth noting that by taking $\alpha = 0, g_m = 1, a_p = h, c_q = h$, we get a result which is essentially the sixth theorem of Wimp and Luke [6], which in its turn contains as a particular case Theorem 6 of Meijer [4].

One can easily extend a result due to Carlitz and Alsalam [7] to G -functions by using the above technique.

Added in proof. In (3.1) taking $a_1 = t$ and then letting $t \rightarrow 0$, we get the sum of an infinite series of G -functions in terms of products of gamma functions, a result which also generalises the known formula due to MacRobert and Ragab [*Math. Z.*, v. 78, 1962] and is perhaps a solitary example of its type for the G -functions.

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* The conditions on the parameters arise due to the particular method followed and can be waived off by analytic continuation.