Computation of Moments of $K_\nu(t)/I_\nu(t)$*

By Jerry Allan Roberts

During the course of some recent research [3] it became necessary to compute certain values of $M_n^{(1)}$, where

$$(1) \qquad M_n^{(\nu)} \equiv \int_0^\infty t^n \frac{K_\nu(t)}{I_\nu(t)}\, dt.\dagger$$

Values of a related function,

$$(n!)^{-2} I(2n) \equiv \frac{2n+1}{(n!)^2} \int_0^\infty t^{2n} \frac{K_1(t)}{I_1(t)}\, dt,$$

had been tabulated by Smythe [5]. These values were used in checking the computed values. When disagreements were encountered, it was decided to investigate the problem more thoroughly. Smythe has since corrected the errors discovered in [5] (see [7]). Smythe's corrected values are in agreement with the values tabulated herein. Smythe also has tabulated values of a function related to $M_{2n}^{(0)}$. These values first were given (with some errors) in [4] and subsequently were corrected in [6]. Another related function

$$H_k = \frac{(-1)^k}{(2k)!} \frac{2}{\pi} \int_0^\infty t^{2k} \frac{K_1(t)}{I_1(t)}\, dt,$$

is tabulated, for $k = 1(1)12$, by Brenner and Sonshine [1]. A comparison of the values given in [1] with those tabulated herein reveals agreement to seven significant digits in most entries, although the entry for $k = 9$ (and $n = 2k = 18$) differs by five units in the sixth significant digit. The author recently has obtained a copy of a report by Haberman and Harley [2] in which values of the function defined in (1), with $\nu = 1$, are tabulated for $n = 1(1)20$. These values, for $n = 2(1)17$, agree to four significant digits with those tabulated herein, as do the values for $n = 18(1)20$ after removal of an extraneous factor of 10. However the value tabulated in [2] for $n = 1$ is completely in error.

It is observed that convergence of (1) requires that $n > 2|\nu| - 1$. If n satisfies this requirement, integration by parts in (1) with

$$u = K_\nu(t)/I_\nu(t), \qquad dv = t^n\, dt$$

and use of the Wronskian

$$(2) \qquad I_\nu(t)K_\nu'(t) - I_\nu'(t)K_\nu(t) = -\frac{1}{t}$$

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† The notation is that of J. C. Cooke and C. J. Tranter, "Dual Fourier-Bessel series", *Quart. J. Mech. Appl. Math.*, v. 12, 1959, 379–386.

(see Watson [9], p. 80) yields

$$M_n^{(\nu)} = \frac{1}{n+1} \int_0^\infty \frac{t^n \, dt}{I_\nu^2(t)}. \tag{3}$$

If n is even, this integral can be evaluated by a method due to Watson [8].

It is observed that if ν is a non-negative integer, (3) is much better suited for numerical integration than is (1), whose integrand has a logarithmic singularity in the higher derivatives at the origin.

Investigation of the integrand of (1) reveals that for large values of n the integrand attains its maximum value near $n/2$. Thus, as n increases, the significant portion of the integrand occurs at increasingly larger values of t. This suggests that, if n is large enough, the use of a few terms of the asymptotic approximation for $K_\nu(t)$ and $I_\nu(t)$ in (1) might yield a reasonable approximation of $M_n^{(\nu)}$. Such an "asymptotic" expansion is purely formal, and a rigorous analysis of its approximative properties has not been feasible thus far. However, in the case $\nu = 1$, the expansion so obtained has been verified numerically and was found to produce results correct to eight significant digits for $n \geqq 22$. The expansion, for $\nu = 1$, was obtained by division of the asymptotic expansion for $K_1(t)$ by that for $I_1(t)$ (see Watson [9], pp. 202, 203), to obtain

$$\frac{K_1(t)}{I_1(t)} \sim \pi e^{-2t} \left[1 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_{12}}{t^{12}} \right]. \tag{4}$$

The values of $a_1, a_2, \cdots, a_{12}$ are given in Table 1. Multiplication of (4) by t^n and formal integration over $(0, \infty)$ yields

$$M_n^{(1)} \sim \frac{\pi}{2} \frac{n!}{2^n} \left[1 + \frac{b_1}{n} + \frac{b_2}{n(n-1)} + \cdots + \frac{b_{12}}{n(n-1)\cdots(n-11)} \right]. \tag{5}$$

The resulting values of $b_1, b_2, \cdots, b_{12}$ also are given in Table 1. A different ex-

TABLE 1

Coefficients of the asymptotic expansions for $K_1(t)/I_1(t)$ and for $M_n^{(1)}$

n	a_n	b_n
1	7.5 0000 0000 (−1)*	1.5 0000 0000 (0)
2	2.8 1250 0000 (−1)	1.1 2500 0000 (0)
3	3.9 8437 5000 (−1)	3.1 8750 0000 (0)
4	2.5 9277 3438 (−1)	4.1 4843 7500 (0)
5	8.3 6059 5703 (−1)	2.6 7539 0625 (1)
6	6.3 3499 1455 (−1)	4.0 5439 4531 (1)
7	4.9 9215 3168 (0)	6.3 8995 6055 (2)
8	3.8 6544 8356 (0)	9.8 9554 7791 (2)
9	6.2 7828 9583 (1)	3.2 1448 4266 (4)
10	4.8 1743 2137 (1)	4.9 3305 0508 (4)
11	1.3 3709 0455 (3)	2.7 3836 1252 (6)
12	1.0 1725 4727 (3)	4.1 6667 5362 (6)

* The number in parenthesis indicates the power of 10 by which the tabulated value is to be multiplied, e.g. $a_1 = 7.5\,0000\,0000 \times 10^{-1} = .75\,0000\,0000$.

TABLE 2
Values of $M_n^{(1)}$

n	$M_n^{(1)}$	n	$M_n^{(1)}$	n	$M_n^{(1)}$
1	∞	35	4.931 2437 (29)	69	4.653 7548 (77)
2	2.503 2970 (0)	36	8.865 6079 (30)	70	1.628 3076 (79)
3	2.330 2884 (0)	37	1.638 2799 (32)	71	5.778 7448 (80)
4	3.771 3888 (0)	38	3.109 3932 (33)	72	2.079 7369 (82)
5	8.391 9202 (0)	39	6.057 1486 (34)	73	7.588 8702 (83)
6	2.343 0922 (1)	40	1.210 2593 (36)	74	2.807 1013 (85)
7	7.817 4119 (1)	41	2.478 7519 (37)	75	1.052 3781 (87)
8	3.022 8568 (2)	42	5.200 8251 (38)	76	3.997 9831 (88)
9	1.326 5732 (3)	43	1.117 2449 (40)	77	1.538 8285 (90)
10	6.505 9571 (3)	44	2.455 9824 (41)	78	5.999 9304 (91)
11	3.523 8109 (4)	45	5.521 7583 (42)	79	2.369 3949 (93)
12	2.088 0460 (5)	46	1.269 0808 (44)	80	9.475 3278 (94)
13	1.343 2135 (6)	47	2.980 2638 (45)	81	3.836 6185 (96)
14	9.320 3281 (6)	48	7.147 8618 (46)	82	1.572 6580 (98)
15	6.937 8268 (7)	49	1.750 1058 (48)	83	6.525 0906 (99)
16	5.514 1223 (8)	50	4.372 5778 (49)	84	2.739 9478 (101)
17	4.660 2384 (9)	51	1.114 3496 (51)	85	1.164 2330 (103)
18	4.173 0262 (10)	52	2.895 6656 (52)	86	5.005 1735 (104)
19	3.946 5249 (11)	53	7.669 3264 (53)	87	2.176 8135 (106)
20	3.930 5825 (12)	54	2.069 6301 (55)	88	9.576 1013 (107)
21	4.112 0667 (13)	55	5.688 6012 (56)	89	4.260 5482 (109)
22	4.508 3159 (14)	56	1.592 0308 (58)	90	1.916 8873 (111)
23	5.168 9385 (15)	57	4.535 1507 (59)	91	8.720 2386 (112)
24	6.185 6175 (16)	58	1.314 5957 (61)	92	4.010 5905 (114)
25	7.712 4276 (17)	59	3.876 3538 (62)	93	1.864 5973 (116)
26	1.000 2730 (19)	60	1.162 4124 (64)	94	8.762 1026 (117)
27	1.347 4500 (20)	61	3.543 9018 (65)	95	4.161 2991 (119)
28	1.882 6477 (21)	62	1.098 1730 (67)	96	1.997 0948 (121)
29	2.724 7472 (22)	63	3.457 9141 (68)	97	9.684 3483 (122)
30	4.080 0104 (23)	64	1.106 1201 (70)	98	4.744 5813 (124)
31	6.313 7304 (24)	65	3.593 5919 (71)	99	2.348 2044 (126)
32	1.008 6574 (26)	66	1.185 4700 (73)	100	1.173 9242 (128)
33	1.661 9034 (27)	67	3.969 9752 (74)		
34	2.821 4328 (28)	68	1.349 3465 (76)		

pansion, obtained by the substitution of the asymptotic expansion for $[I_1(t)]^{-2}$ into equation (3), is given in [1]. This expansion and the one given in equation (5) above are equivalent for large values of n.

A table of values for $M_n^{(1)}$, for $n = 1, 2, \cdots, 100$ is given in Table 2. The values for $n = 2, 3, \cdots, 25$ were obtained by numerical integration on an IBM 1410 computer. For this integration (3) was used, and the interval of integration was mapped into a finite interval by a simple change of variable. The resulting integral was evaluated, using Simpson's rule, with the increment chosen to assure that the truncation error in the final result would be negligible to eight significant digits. As a further check on the results so obtained, this numerical integration later was repeated with a smaller increment. In no case did the two resulting values differ by more than one unit in the eighth significant digit. The values of $M_n^{(1)}$ for $n = 20$, 21, $\cdots$, 100 were computed by use of the "asymptotic" expansion given in (5). The "overlap" of the two methods for $n = 20, 21, \cdots, 25$ was provided as a numerical verification of this "asymptotic" expansion. It was found that these two methods, gave results which differed (in the eighth significant digit) by thirteen units for $n = 20$, four units for $n = 21$, and not more than one unit for $n = 22, 23, 24$, and 25.

Since the numerical integrations were computed in an ascending order, i.e., the integrand for $M_n^{(1)}$ was multiplied by t in order to obtain the integrand for $M_{n+1}^{(1)}$, $n = 2, 3, \cdots, 24$, and, in view of the agreement indicated above, it is felt that all values in Table 2 are correct, except for possible rounding errors of one unit in the eighth significant digit.

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Bessel-Function Identities Needed for the Theory of Axisymmetric Gravity Waves

By Lawrence R. Mack

1. Introduction. Certain identities involving integrals of products of Bessel functions are required for analyses of finite-amplitude axisymmetric gravity waves [3], [4]. The specific identities needed through the third-order wave solution are of two distinct types. The first type equates to zero the sum of two or three integrals of products of several Bessel functions, all integrands in a particular identity being products of the same number of Bessel functions. Of the required identities of this type the one with products of two Bessel functions is trivial, while those whose integrands are products of three Bessel functions are obtainable from the results of Fettis [1]. Each identity of the second type equates an integral of the product of four Bessel functions to the sum of an infinite number of products of pairs of

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