TECHNICAL NOTES AND SHORT PAPERS

A Note on a Theorem of J. N. Franklin

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Let $x^{(0)} = x$ be a d-dimensional real column vector and consider the sequence of d-dimensional vectors $x^{(n)}$ $(n = 0, 1, 2, \cdots)$ defined by the relation

$$x^{(n+1)} \equiv Ax^{(n)} + b \pmod 1, \tag{1}$$

where A is a $d \times d$ matrix with real components and b is a fixed d-dimensional real column vector. In a recent paper [1] J. N. Franklin proved the following

THEOREM. *If all the components of the matrix A are rational integers, then the sequence of d-dimensional vectors $x^{(n)}$ $(n = 0, 1, 2, \cdots)$ defined by (1) is equidistributed modulo one for almost all initial vectors x if A has no eigenvalue equal to zero or a root of unity; when $b = 0$, the sequence is equidistributed modulo one for almost all x if and only if A has no eigenvalue equal to zero or a root of unity.*

Franklin's proof of this result makes use of the criterion of H. Weyl [3] for the equidistribution modulo one of sequences of vectors and the individual ergodic theorem due to F. Riesz as well. The purpose of this note is to give a simple proof of the theorem of Franklin on the basis of the criterion of Weyl only.

By the way, we note that the above theorem has some applications in the theory of normal numbers. As a sample of this we may mention the following. J. E. Maxfield [2] introduced the notion of normal d-tuples (or, d-dimensional vectors) with real components to scale r, where $r \geqq 2$ is a fixed integer, and showed that almost all d-tuples are normal to scale r (and hence normal to scale r for every $r \geqq 2$, i.e. absolutely normal). Indeed, a d-tuple x is normal to scale r if and only if the sequence $r^n x$ $(n = 0, 1, 2, \cdots)$ is equidistributed modulo one (cf. [2, Theorem 5]): thus, the conclusion that almost all d-tuples are normal to scale r is an immediate consequence of the theorem of Franklin with an appropriate diagonal matrix A and $b = 0$.

Now, let $x^{(n)}$ $(n = 0, 1, 2, \cdots)$ be a sequence of real d-dimensional vectors defined by (1), where b is a fixed real d-dimensional vector and we suppose that the components of the $d \times d$ matrix A are integers. Let h be a d-dimensional vector with integer components and put

$$F_n(h, x) = \frac{1}{n} \sum_{j=0}^{n-1} e((h, x^{(j)})),$$

where we write $e(t) = \exp 2\pi i t$ and denote by (u, v) the inner product of the vectors u and v. If we set

$$b^{(0)} = 0, \qquad b^{(j)} = b + Ab + \cdots + A^{j-1}b \qquad (j \geqq 1),$$

then

$$F_n(h, x) = \frac{1}{n} \sum_{j=0}^{n-1} e((h, A^j x + b^{(j)})),$$

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so that

$$|F_n(h, x)|^2 = \frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} e((h, A^j x + b^{(j)}))\overline{e((h, A^k x + b^{(k)}))}$$
$$= \frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} e((h, b^{(j)} - b^{(k)}))e((h, (A^j - A^k)x)).$$

We have $(A^j - A^k)^* h \neq 0$ for any $h \neq 0$, provided that A has no eigenvalue which is zero or a root of unity. (The proof of this fact can easily be carried out in a similar way to that of [1, §4, Lemma].) Since $(h, (A^j - A^k)x) = ((A^j - A^k)^* h, x)$, it follows from this that

$$\int_{C^d} e((h, (A^j - A^k)x))\, dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

where C^d is the d-dimensional unit cube,

$$C^d: 0 \leqq x_s < 1 \qquad (s = 1, \cdots, d).$$

Hence we obtain

$$\int_{C^d} |F_n(h, x)|^2\, dx = \frac{1}{n} \tag{2}$$

for any vector $h \neq 0$ with integer components. It is now quite easy to deduce from (2) that we have, for such h,

$$\lim_{n\to\infty} F_n(h, x) = 0 \tag{3}$$

almost everywhere on C^d. In fact, we have

$$\sum_{m=1}^{\infty}\int_{C^d} |F_{m^2}(h, x)|^2\, dx = \sum_{m=1}^{\infty}\frac{1}{m^2} < \infty,$$

and this implies that the series $\sum_{m=1}^{\infty} |F_{m^2}(h, x)|^2$ converges for almost all x in C^d and so

$$\lim_{m\to\infty} F_{m^2}(h, x) = 0$$

for almost all x in C^d. If $m^2 \leqq n < (m+1)^2$, then

$$|F_n(h, x) - F_{m^2}(h, x)| \leqq \frac{2(n - m^2)}{n} < \frac{4}{\sqrt{n}},$$

from which follows (3) at once. (This argument for the deduction of (3) has already been used by Weyl [3, §7].) We thus have proved the theorem via Weyl's criterion, except for the necessity of the condition that A have no eigenvalue equal to zero or a root of unity, in the case of $b = 0$. But, for $b = 0$, we may simply proceed just in the same way as in [1, the latter half of §5] and we shall refer to it for the detail.

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1. J. N. Franklin, "Equidistribution of matrix-power residues modulo one," *Math. Comp.*, v. 18, 1964, pp. 560–568.
2. J. E. Maxfield, "Normal k-tuples," *Pacific J. Math.*, v. 3, 1953, pp. 189–196. MR **14**, 851.
3. H. Weyl, "Über die Gleichverteilung von Zahlen modulo Eins," *Math. Ann.*, v. 77, 1916, pp. 313–352.