A Division Algebra for Sequences Defined on all the Integers

By D. H. Moore

The convolution ring, \( S \), of sequences defined on the nonnegative integers, and the embedding of this ring in a field, have been discussed by Brand [1], Moore [2], [3], Traub [6], and others. Brand [1] specifically mentions that the field in which he embeds \( S \) is a field of ordered pairs of members of \( S \). Traub does not identify his field and does not mention "ordered pairs", but he mentions an analogy to Mikusiński's work [7], and so he probably had in mind the same field of ordered pairs as did Brand. In [2] this writer showed that it was not necessary to create such a field of ordered pairs since there already existed a more natural, less abstract field in which to embed \( S \). It is the purpose of this article to introduce this already existing and more natural field, \( \mathfrak{F} \), in which \( S \) may be embedded.

It will be assumed that the reader is familiar with the convolution algebra of sequences as given in [1], [3], and [6] to the point of recognizing \( S \) as an integral domain in which convolution products defined by

\[
\{a_r\} \{b_r\} = \left\{ \sum_{\mu=0}^{\infty} a_{\mu} b_{r-\mu} \right\}
\]

contain no divisors of zero, in which the multiplicative unity is the sequence

\[ \{1, 0, 0, 0, \ldots, 0, \ldots\} \]

in which sequences of the form

\[ \{c, 0, 0, 0, \ldots, 0, \ldots\} \]

behave like numbers and are identified with numbers:

\[ c = \{c, 0, 0, 0, \ldots, 0, \ldots\} \]

in which the sequence

\[ \{0, 1, 0, 0, 0, \ldots, 0, \ldots\} \]

is a shift operator denoted by "\( \tau \)", in which the sequence

\[ \{1, 1, 1, \ldots, 1, \ldots\} \]

is a summing operator denoted by "\( \sigma \)", and in which members of \( S \) have operational forms in terms of \( \tau \) and/or \( \sigma \).

The sequences \( \sigma \) and \( \tau \) are related by the equation

\[ \sigma (1 - \tau) = 1 \]

and since \( S \) has no divisors of zero we introduce fractions and write (for example)

\[ \sigma = \frac{1}{1 - \tau} \]

\[ \frac{1}{\sigma} = 1 - \tau = \{1, -1, 0, 0, 0, \ldots\} \]


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The fraction $1/\tau$, for example, does not exist as a member of $S$. But $1/\tau$ will exist as a member of $F$.

Let $F$ be the class of number valued sequences defined over the integral domain, $J$, each of which assigns at most a finite number of nonzero values to negative integers. For each member of $F$ there is a least integer, $m$, to which the sequence assigns a nonzero value; the sequence will be said to enter at $m$, and the members of $F$ will be called entering sequences. Equality, sums, and products with numbers, of members of $F$ are defined in the usual termwise way. A suggested notation for such a sequence is

$$\{1, 2, 3 | 4, 5, 6, \ldots \}$$

where the vertical line—playing a role like a decimal point—separates values assigned to the negative integers from values assigned to the nonnegative integers, and the zeros assigned on the left are omitted for convenience.

Let $v$ be a variable on $J$. We define the unit step formula $u(v)$:

$$u(v) = \begin{cases} 0, & v < 0, \\ 1, & v \geq 0. \end{cases}$$

Then $\{u(v)\}$ and $\{(v + 1)u(v)\}$ (for example) are members of $F$, whereas $\{v + 1\}$ is not. The braces serve to bind out "$v$" converting a formula into a notation for a sequence.

There is a natural one-to-one correspondence between $S$ (sequences defined on the nonnegative integers) and the subclass, $F_0$, of $F$ consisting of sequences which enter at nonnegative points:

$$\{a_0, a_1, a_2, \ldots \} \leftrightarrow \{\ldots, 0, 0 | a_0, a_1, a_2, \ldots \}. \label{eq:correspondence}$$

The convolution, or convolution product, of two sequences $a$ and $b$ from $F$ is defined by

$$ab = \left\{ \sum_{\mu=\alpha}^{\infty} a_{\mu} b_{\nu-\mu} \right\}. \label{eq:convolution}$$

If $c$ enters at $\alpha$ or to the right of $\alpha$ and $b$ enters at $\beta$ or to the right of $\beta$, then

$$(ab)_\nu = \begin{cases} \sum_{\mu=\alpha}^{\nu} a_{\mu} b_{\nu-\mu}, & \nu \geq \alpha + \beta, \\ 0, & \nu < \alpha + \beta, \end{cases}$$

$$ab = \left\{ u(\nu - \alpha - \beta) \sum_{\mu=\alpha}^{\nu} a_{\mu} b_{\nu-\mu} \right\}. \label{eq:convolution2}$$

The summation limits are finite in (5) since the sequences are entering sequences. In particular, if $a$ and $b$ are members of the subclass $F_0$, we may take $\alpha = \beta = 0$ in (5) and (5) becomes

$$ab = \left\{ u(\nu) \sum_{\mu=0}^{\nu} a_{\mu} b_{\nu-\mu} \right\}. \label{eq:convolution3}$$

A comparison of (1) and (6) shows that the correspondence (2) is an isomorphism under convolution; we embed $S$ in $F$, identify $S$ with $F_0$, elevate (2) to an...
equality, and permit any notation for a member of $S$ to be used as a notation for the corresponding member of $S_0$. In particular

$$1 = \{\cdots, 0, 0, 0 | 1, 0, 0, 0, \cdots, 0, \cdots\}$$

$$\sigma = \{\cdots, 0, 0, 0 | 1, 1, 1, \cdots, 1, \cdots\}$$

$$\tau = \{\cdots, 0, 0, 0 | 0, 1, 0, 0, 0, \cdots, 0, \cdots\}$$

$$\tau^m = \{\cdots, 0, 0, 0 | 0, 0, \cdots, 0, 1, 0, 0, 0, \cdots\} \quad m \text{ zeros}$$

Defining $\xi$ by:

$$\xi = \{1 | 0, 0, 0, \cdots, 0, \cdots\}$$

we have

$$\xi^m = \left\{1, 0, 0, 0, \cdots, 0 | 0, 0, 0, \cdots, 0, \cdots\right\} \quad m \text{ digits}$$

Equations (7) and (9) may be verified by induction. Using (5) we may verify that

$$\tau^m = 1$$

$$\tau^m + n = \xi^m \tau^n$$

$$\xi^m + n = \xi^m \xi^n \quad m, n \text{ positive integers.}$$

$$\tau^m[a_r] = \{a_{r-m}\}$$

$$\xi^m[a_r] = \{a_{r+m}\}$$

Under ordinary addition and convolution multiplication $\mathcal{F}$ is a field. We need only verify here that each nonzero member of $\mathcal{F}$ has a multiplicative inverse. To begin with, every sequence of the form

$$\{a_0, a_1, a_2, \cdots\}$$

in which $a_0 \neq 0$ (the sequence enters at the origin) has an inverse:

$$\{x_0, x_1, x_2, \cdots\}$$

which may be evaluated as follows:

$$\{a_0, a_1, a_2, \cdots\} \{x_0, x_1, x_2, \cdots\} = \{1, 0, 0, 0, \cdots\}$$

$$a_0 x_0 = 1$$

$$a_0 x_1 + a_1 x_0 = 0$$

$$a_0 x_2 + a_1 x_1 + a_2 x_0 = 0$$

$$\vdots$$

Since the only division involved in solving for the $x$'s is division by $a_0$, and $a_0 \neq 0$, the $x$'s exist and so the desired inverse exists.

Finally, let $a$ be any nonzero member of $\mathcal{F}$ which does not enter at the origin.
Since \( a \) is an entering sequence, there exists a sequence \( A \) and a positive integer \( m \) such that either

\[
(10) \quad a = \tau^m A \quad \text{or} \quad a = \zeta^m A
\]

where \( A \) enters at the origin, and so has an inverse \( A^{-1} \) by the preceding paragraph. Then either

\[
(A^{-1}\tau^m)a = 1 \quad \text{or} \quad (A^{-1}\zeta^m)a = 1
\]

and so, in any case, \( a \) has a multiplicative inverse, and \( \mathcal{F} \) is a field.

Since \( \mathcal{F} \) contains no divisors of zero, products lead to the introduction of fractions:

\[
\begin{align*}
& a, b, c \in \mathcal{F} \\
& \quad \text{and} \\
& ab = c \\
& \quad \text{and} \\
& a \neq 0 \\
& \quad \text{and} \\
& \frac{c}{a} \text{ exists as a member of } \mathcal{F} \\
& \quad \text{and} \\
& \frac{c}{a} = b \\
& \quad \text{and} \\
& a \left( \frac{c}{a} \right) = c.
\end{align*}
\]

In particular

\[
\zeta = \{1, 0, 0, 0, \ldots\} = \frac{1}{\tau} = \frac{\{1, 0, 0, 0, \ldots\}}{\{0, 1, 0, 0, 0, \ldots\}}
\]

and \( 1/\tau \) exists as a member of \( \mathcal{F} \).

Members of \( \mathcal{F} \) may be put into operational form in terms of \( \sigma, \tau, \) and/or \( \zeta \).

**Example 1.**

\[
\left\{ \frac{\nu(\nu - 1)}{2} u(\nu + 2) \right\} = \{3, 1 | 0, 0, 0, 1, 3, 6, 10, 15, \ldots\}
\]

\[
= \zeta^2 \left\{ \frac{\nu(\nu - 2)}{2} u(\nu) \right\} = \frac{\zeta^2}{2} \left\{ (\nu^2 - 5\nu + 6)u(\nu) \right\}
\]

\[
= \frac{\zeta^2}{2} \left( \sigma^2 \tau + 2\sigma^3 \tau^2 - 5\sigma^2 \tau + 6\sigma \right)
\]

where \( \nu u(\nu) = \sigma^2 \tau \) and \( \nu^2 u(\nu) = \sigma^2 \tau + 2\sigma^3 \tau^2 \) as shown in [2], and as may be checked straightforwardly. Then

\[
\left\{ \frac{\nu(\nu - 1)}{2} u(\nu + 2) \right\} = \sigma^3 - 2\sigma^2 \zeta + 3\sigma \zeta^2.
\]

In Traub [6, p. 196], every quotient of "generalized" sequences with a nonzero denominator equals a shift operator times an ordinary sequence. Thus, in Traub’s notation,

\[
\frac{f}{g} = \frac{f}{\omega e} = \frac{f}{e}
\]
where \( f/e \) equals an ordinary sequence since \( e \) assigns a nonzero value to the origin; \( \omega^{-t} \) is a shift operator, and is a "generalized" sequence—an ordered pair of ordinary sequences. In comparison, in the present paper, we are dealing with entering sequences (defined on \( J \)) instead of ordered pairs, and every quotient, \( b/a \), of entering sequences (with nonzero denominator) equals an entering sequence. In evaluating \( b/a \) we may replace \( a \), as in (10), by \( \tau^n A \) or \( \zeta^n A \), as appropriate, and obtain respectively

\[
\frac{b}{a} = \tau^n \frac{b}{A} \quad \text{or} \quad \frac{b}{a} = \tau^m \frac{b}{A}
\]

where \( b/A, \tau^m \), and \( \tau^n \) are all entering sequences.

**Example 2.**

\[
\begin{align*}
\{1, -1, 1, -1, 1, -1, \ldots\} &= \frac{1}{1 + \tau} \frac{1}{1 - \tau} \\
\{1, 1, 1, 1, 1, 1, \ldots\} &= \frac{\tau}{1 + \tau} \\
&= \tau \{1, -2, 2, -2, 2, -2, \ldots\} \\
&= \{0, 0, 0, 1, -2, 2, -2, 2, -2, \ldots\}.
\end{align*}
\]

**Example 3.**

\[
\begin{align*}
\{3, 1 \mid 0, 0, 1, 3, 6, 10, 15, \ldots\} &- \{0, 0, 0, 1, 3, 6, 10, 15, \ldots\} \\
&= \frac{\sigma^3 - 2\sigma^2 \zeta + 3\sigma \zeta^2}{\tau^3(\tau + 1)^3} \\
&= \frac{1}{(1 - \tau)^3} - 2\frac{1}{(1 - \tau)^2} \frac{1}{\tau} + 3\frac{1}{1 - \tau} \frac{1}{\tau^2} \\
&= \frac{1}{(1 - \tau)^3} - 2\frac{1}{(1 - \tau)^2} \frac{1}{\tau} + 3\frac{1}{1 - \tau} \frac{1}{\tau^2} \\
&= \frac{(6 \tau - 8 \tau^2 + 3 \tau^3)}{(1 - \tau)^3} \\
&= \{6 \tau - 8 \tau^2 + 3 \tau^3\} \{1, 0, 3, 0, 6, 0, 10, 0, 15, 0, \ldots\} \\
&= \{6, 0, 18 \mid 0, 36, 0, 60, \ldots\} \\
&+ \{-8, 0, -24, 0 \mid -48, 0, -80, 0, \ldots\} \\
&+ \{3, 0, 9, 0, 18 \mid 0, 30, 0, 45, \ldots\} \\
&= \{3, -8, 15, -24, 36 \mid -48, 66, -80, 105, \ldots\}.
\end{align*}
\]

The last result may be checked by cross multiplication:

\[
\begin{align*}
\{0, 0, 0, 1, 3, 3, 1, 0, 0, 0, \ldots\} &\{3, -8, 15, -24, 36 \mid -48, 66, -80, 105, \ldots\} \\
&= \{3, 1 \mid 0, 0, 1, 3, 6, 10, 15, \ldots\}.
\end{align*}
\]

A convenient way to multiply two entering sequences is to ignore the vertical lines at first, and then insert a vertical line in the final answer, following rules similar to those for the insertion of a decimal point in a product of decimals.
George Boole’s operator, $E$, [4, p. 16] which shifts a sequence to the left and replaces by zero the terms which pass the origin, operates only on sequences which vanish to the left of the origin:

$$E^n[f(v)u(v)] = [f(v + n)u(v)], \quad n = \text{nonnegative integer.}$$

Thus $E$ cannot be identified with $\xi$; neither is $E$ to be discarded, since there is no convolution product to do the job that $E$ does, and that job is important. However, George Boole’s symbolic method [4, p. 215] is salvaged if $E$ is replaced by $\xi$ as discussed in [2]. Thus, Boole’s symbolic equation [4, pp. 217, 218]

$$[b^x] - a = [b^x] - a + ca^x, \quad c = \text{arbitrary constant } a, b \text{ numbers}$$

becomes:

$$\frac{b^x}{b - a} + \frac{ca^x}{a - b} = \frac{b^x}{b - a} + \frac{a^x}{a - b}.$$

This follows from the equation

$$\frac{b^x}{b - a} = \frac{1}{b - a} = \frac{1}{\xi - c}, \quad c = \text{number}$$

which is easily checked by cross multiplication. To prove (11) we have

$$\frac{b^x}{b - a} = \frac{\xi}{\xi - a} = \frac{1}{\xi - c} = \frac{1}{\xi - b} + \frac{1}{a - b} \frac{\xi}{\xi - a}.$$

When operational forms of sequences are expressed in terms of $\xi$ they match the $Z$-transforms of sequences as used, for example, by Aseltine [5] (hence the use of “$\xi$” for the reciprocal of $\tau$). For example [5, p. 259]

$$\{u(\nu)\} = \sigma = \frac{1}{\xi - 1}.$$

But now $\xi$ is a sequence and not a variable, a formula in $\xi$ equals a sequence rather than being a “transform” of it, and the introduction of the $\xi$-forms requires no theory of convergence of power series. In [2, pp. 140–143] it is shown that results previously obtained using the theory of functions of a complex variable including branch cuts and the theory of residues, may be obtained by purely algebraic methods from the field properties of $\xi$.

California State Polytechnic College
Pomona, California


