# A Note on an Iterative Method for Generalized Inversion of Matrices*

By Adi Ben-Israel

The iterative method of Schulz [4], [3] for matrix inversion was generalized in [1] as follows:

THEOREM 1. *The sequence of matrices defined by*

$$X_{k+1} = X_k(2P_{R(A)} - AX_k) \qquad (k = 0, 1, \cdots) \tag{1}$$

*where $X_0$ is an $n \times m$ complex matrix satisfying*

$$X_0 = A^*B_0\,, \quad B_0 \text{ some nonsingular } m \times m \text{ matrix}, \tag{2}$$

$$X_0 = C_0A^*, \quad C_0 \text{ some nonsingular } n \times n \text{ matrix}, \tag{3}$$

$$\| AX_0 - P_{R(A)} \| < 1, \quad (\| \ \| \text{ any matrix norm [3]}), \tag{4}$$

$$\| X_0A - P_{R(A^*)} \| < 1, \tag{5}$$

*converges to the generalized inverse $A^+$ of $A$.*

As pointed out in [1], the computational significance of the method (1) is limited by the need for knowledge of $P_{R(A)}$ (and of $P_{R(A^*)}$ if condition (5) is to be checked). This difficulty is evaded in the following theorem.

THEOREM 2. *Let $A$ be an arbitrary (nonzero) complex $m \times n$ matrix of rank $r$ and let*

$$\lambda_1(AA^*) \geqq \lambda_2(AA^*) \geqq \cdots \geqq \lambda_r(AA^*)$$

*denote the nonzero eigenvalues of $AA^*$. If the real scalar $\alpha$ satisfies*

$$0 < \alpha < \frac{2}{\lambda_1(AA^*)} \tag{6}$$

*then the sequence defined by:*

$$X_0 = \alpha A^* \tag{7}$$

$$X_{k+1} = X_k(2I - AX_k) \qquad (k = 0, 1, \cdots)\,. \tag{8}$$

*converges to $A^+$ as $k \to \infty$.*

*Proof.* $X_0$ defined by (7), (6) satisfies (2), (3), (4) and (5). To prove that $X_0$ of (7), (6) satisfies (4) we note that $AA^+$ $(=P_{R(A)})$ and $AA^*$ are commuting Hermitian matrices with the same range space. The eigenvalues of the $m \times m$ matrix: $AX_0 - P_{R(A)} = \alpha AA^* - AA^+$ are therefore

$$\begin{cases} \alpha\lambda_i(AA^*) - 1 & (i = 1, \cdots, r) \\ 0 & (i = r + 1, \cdots, m) \end{cases} \tag{9}$$

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and, by (6), are all: $<1$ in absolute value:

$$|\lambda_i(\alpha AA^* - AA^+)| < 1 \qquad (i = 1, \cdots, m) \tag{10}$$

similarly

$$|\lambda_i(\alpha A^*A - A^+A)| < 1 \qquad (i = 1, \cdots, n). \tag{11}$$

(Indeed the nonzero eigenvalues of $(\alpha AA^* - AA^+)$, $(\alpha A^*A - A^+A)$ are identical.) With the $\text{lub}_S$-norm [3, p. 44] in (4) and (5), both hold because of (10) and (11). (Actually (10) and (11) suffice for the convergence of (8).)

Now the process (1) initiated with: $X_0 = \alpha A^*$ retains the form [1, Eq. (12)]:

$$X_k = C_k A^* \qquad (k = 1, 2, \cdots) \tag{12}$$

and since

$$A^* P_{R(A)} = A^* \tag{13}$$

it follows that:

$$X_k(2P_{R(A)} - AX_k) = X_k(2I - AX_k) \qquad (k = 0, 1, \cdots) \tag{14}$$

and the convergence of (8) follows from that of (1). Q.E.D.

*Remarks.*

a) Similarly, the sequence defined by

$$X_{k+1} = (2I - X_k A)X_k \qquad (k = 0, 1, \cdots) \tag{15}$$

with $X_0 = \alpha A^*$, converges to $A^+$.

b) In using the method (8) it is not necessary to compute $\lambda_1(AA^*)$: Writing

$$AA^* = (b_{ij}) \qquad (i, j = 1, \cdots, m)$$

we conclude from the Gershgorin theorem, [3] that:

$$\lambda_1(AA^*) \leqq \max_{i=1,\cdots,m} \left\{ \sum_{j=1}^{m} |b_{ij}| \right\}.$$

Condition (6) can therefore be replaced, e.g. by

$$0 < \alpha < \frac{2}{\max_{i=1,\cdots,m} \left\{ \sum_{j=1}^{m} |b_{ij}| \right\}}. \tag{16}$$

c) Examples and applications will be given in [2].

Technicon, Israel Institute of Technology
Haifa, Israel

Northwestern University
Evanston, Illinois

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