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On the First Positive Zero of $P_{\nu-1/2}^{(-m)}(\cos\theta)$, Considered as a Function of ν

By R. D. Low

1. Introduction. Several years ago Pal [1], [2] published two papers in which he considered the roots of the equations $P_\nu^{(m)}(\mu) = 0$ and $(d/d\mu)\, P_\nu^{(m)}(\mu) = 0$ regarded as equations in ν.† In these equations m is an integer and $\mu = \cos\theta$. Among the roots which Pal computed and tabulated are those of the equation $P_\nu^{(2)}(\cos\theta) = 0$ for $\theta = \pi/12$, $\pi/6$, and $\pi/4$, and he lists as the first root in each case: 4.77, 2.26, and 1.52. In view of the fact that $P_\nu^{(2)}(\cos\theta) = \nu(\nu+2)(\nu^2-1)\cdot P_\nu^{(-2)}(\cos\theta)$, it must be assumed that the numbers just mentioned are respectively the first positive roots of the equation $P_\nu^{(-2)}(\cos\theta) = 0$ for $\theta = \pi/12$, $\pi/6$, and $\pi/4$, since the equation $P_\nu^{(2)}(\cos\theta) = 0$ has the roots -2, -1, 0, and 1 regardless of the value of θ. In any event it will be seen that the numbers 4.77, 2.26, and 1.52 are not roots at all in as much as they are *less than* the first element of a sequence of lower bounds to be exhibited below.

2. A Sequence of Lower Bounds. We restrict our attention to the function $P_{\nu-1/2}^{(-m)}(\cos\theta)$ in which $m = 1, 2, 3, \cdots$ because of the identity [3]

$$P_{\nu-1/2}^{(m)}(\cos\theta) = (-1)^m(\nu^2 - \tfrac{1}{4})(\nu^2 - \tfrac{9}{4}) \cdots [\nu^2 - (2m-1)^2/4]P_{\nu-1/2}^{(-m)}(\cos\theta),$$

which shows that the zeros of $P_{\nu-1/2}^{(m)}(\cos\theta)$ consist of $\pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm(m-\frac{1}{2})$, together with those of $P_{\nu-1/2}^{(-m)}(\cos\theta)$. It is known that $P_{\nu-1/2}^{(-m)}(\cos\theta)$, considered as a function of the complex variable ν, has infinitely many zeros which are all real and simple. Moreover, since $P_{\nu-1/2}^{(-m)}(\cos\theta)$ is an even function of ν which does not vanish for $\nu = 0$, only its positive zeros need be considered. Hence the purpose of the present investigation is to establish a sequence of lower bounds for the first positive zero of $P_{\nu-1/2}^{(-m)}(\cos\theta)$. In addition to the properties mentioned already, it is also known that $P_{\nu-1/2}^{(-m)}(\cos\theta)$ is an entire function of order unity. Hence if $\nu_{n,m}(\theta)$ denotes its nth positive zero, $P_{\nu-1/2}^{(-m)}(\cos\theta)$ can be expressed as an infinite product of the form

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† A trivial change in notation has been made; Pal uses n instead of ν.

$$(1) \qquad P_{\nu-1/2}^{(-m)}(\cos\theta) = P_{-1/2}^{(-m)}(\cos\theta)\prod_{n=1}^{\infty}\left\{1-\frac{\nu^2}{\nu_{n,m}^2(\theta)}\right\}.$$

On the other hand we may also write [3, p. 60]

$$(2) \qquad P_{\nu-1/2}^{(-m)}(\cos\theta) = \frac{\tan^m\theta/2}{m!}\,{}_2F_1(\tfrac12+\nu,\tfrac12-\nu;m+1;\sin^2\theta/2),$$

and by combining (1) and (2), we obtain

$$(3) \qquad \prod_{n=1}^{\infty}\left\{1-\frac{\nu^2}{\nu_{n,m}^2(\theta)}\right\} = \frac{{}_2F_1(\tfrac12+\nu,\tfrac12-\nu;m+1;\sin^2\theta/2)}{{}_2F_1(\tfrac12,\tfrac12;m+1;\sin^2\theta/2)}.$$

If we set $\zeta = \sin^2\theta/2$ it is not difficult to show that the right side of (3) can be written in the form

$$(4) \qquad \sum_{l=0}^{\infty} a_{l,m}(\theta)\nu^{2l},$$

where

$$(5) \qquad a_{l,m}(\theta) = \frac{\displaystyle\sum_{k=l}^{\infty}\frac{b_{l,k}\zeta^k}{(m+1)_k\,k!}}{{}_2F_1(\tfrac12,\tfrac12;m+1;\zeta)},$$

and the $b_{l,k}$ are such that

$$(6) \qquad (\tfrac12+\nu)_k(\tfrac12-\nu)_k = \sum_{l=0}^{k} b_{l,k}\nu^{2l}, \qquad b_{0,0}=1.$$

Next if we denote the left side of (3) by $f(\nu^2)$, take the logarithmic derivative (with respect to ν^2), multiply the result by $f(\nu^2)$, differentiate $l-1$ times, set $\nu=0$, and realize that $f^{(l)}(0) = l!a_{l,m}$, we find

$$(7) \qquad \sum_{p=1}^{l} S_{p,m}(\theta)a_{l-p,m}(\theta) = -la_{l,m}(\theta), \qquad l = 1, 2, 3, \cdots,$$

where

$$(8) \qquad S_{p,m}(\theta) = \sum_{n=1}^{\infty}\nu_{n,m}^{-2p}(\theta).$$

The desired sequence of lower bounds for $\nu_{1,m}(\theta)$ follows directly from (7) and (8). Indeed from (8) we have $\nu_{1,m}^{-2p}(\theta) < S_{p,m}(\theta)$, and if we denote $[S_{p,m}(\theta)]^{-1/2p}$ by $\lambda_m^{(p)}(\theta)$, then

$$(9) \qquad \nu_{1,m}(\theta) > \lambda_m^{(p)}(\theta), \qquad p = 1, 2, 3, \cdots.$$

It is a simple matter to solve (7) for the S's and we then find for the first three λ's:

$$\lambda_m^{(1)} = [-a_{1,m}]^{-1/2},$$

$$\lambda_m^{(2)} = [a_{1,m}^2 - 2a_{2,m}]^{-1/4},$$

$$\lambda_m^{(3)} = [-a_{1,m}^3 + 3a_{1,m}a_{2,m} - 3a_{3,m}]^{-1/6}.$$

It is thus evident that the elements of the sequence $\{\lambda_m^{(p)}(\theta)\}$ depend upon the

$a_{l,m}$ and these coefficients in turn depend upon the $b_{l,k}$ according to (5). From (6) it is obvious that $b_{0,k} = [(\frac{1}{2})_k]^2 = 1^2 \cdot 3^2 \cdots (2k-1)^2/4^k$, and by straightforward calculations we find

$$b_{1,k} = -4H_{1,k}b_{0,k},$$
$$b_{2,k} = 8(H_{1,k}^2 - H_{2,k})b_{0,k},$$
$$b_{3,k} = -\tfrac{32}{3}(H_{1,k}^3 - 3H_{1,k}H_{2,k} + 2H_{3,k})b_{0,k},$$

where

$$H_{p,k} = \sum_{n=1}^{k} (2n-1)^{-2p}, \qquad p = 1, 2, 3, \cdots.$$

3. Some Numerical Results and Comments. In this section we record, in the table below, the results of some computations performed on a desk calculator in the case $m = 2$. This value of m has been chosen: (i) to illustrate the procedure outlined in the previous section, and (ii) to point out the discrepancy mentioned in the introduction regarding the first roots of the equation $P_\nu^{(2)}(\cos\theta) = 0$ as calculated by Pal. Only the elements $\lambda_2^{(1)}(\theta)$ and $\lambda_2^{(2)}(\theta)$ of the sequence $\{\lambda_m^{(p)}(\theta)\}$ have been calculated primarily because of the problem of significant figures for larger values of p. Also the element $\lambda_2^{(1)}(\theta)$ is already sufficient to bring out the erroneous nature of the "first roots" mentioned above.

θ	Pal	$\lambda_2^{(1)}(\theta)$	$\lambda_2^{(2)}(\theta)$	$\nu_{1,2}^{(a)}(\theta)$
$\pi/12$	5.27	13.24	18.76	19.79
$\pi/6$	2.76	6.64	9.42	9.96
$\pi/4$	2.02	4.45	6.32	6.70
$\pi/3$	—	3.36	4.79	5.08
$5\pi/12$	—	2.71	3.98	4.13
$\pi/2$	—	2.29	3.31	3.50

In the column headed "Pal", the entries are Pal's first roots corrected by the additive factor $\frac{1}{2}$ which is necessary because in his equation the degree of the Legendre function is ν rather than $\nu - \frac{1}{2}$. With the exception of $\nu_{1,2}(\pi/2) = 3.50$, which is an exact value, the entries in the column headed $\nu_{1,2}^{(a)}(\theta)$ are the values of $\nu_{1,2}(\theta)$ as computed from the first two terms in the asymptotic expansion

$$\nu_{n,m}(\theta) = \left(n - \frac{1}{4} + \frac{m}{2}\right)\frac{\pi}{\theta} - \frac{(4m^2-1)\cot\theta}{8\theta[1 + (n - 1/4 + m/2)\pi/\theta]} + O(n^{-2}),$$

which was derived from [3, p. 71].

Although no claim is made to the effect that the sequence $\{\lambda_m^{(p)}(\theta)\}$ even converges, let alone that it converges to the true value of $\nu_{1,2}(\theta)$; the above table suggests that this may be the case at least for $m = 2$. Along these lines it is perhaps worth mentioning, for example, that the function $\cos\pi\nu$, like $P_{\nu-1/2}^{(-m)}(\cos\theta)$, is: even in ν, entire of order unity, and has infinitely many zeros which are all real and simple. For $\cos\pi\nu$ the coefficients, analogous to the $a_{l,m}(\theta)$ in (4), are $(-1)^l\pi^{2l}/(2l)!$, and with considerably less effort than was required in the case of the Legendre function one finds $\lambda^{(1)} = 0.45016$, $\lambda^{(2)} = 0.49818$, $\lambda^{(3)} = 0.49988$, etc. The convergence of the sequence $\{\lambda^{(p)}\}$ to the true value $\nu_1 = \frac{1}{2}$ is strongly suggested.

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Some Integrals of Ramanujan and Related Contour Integrals*

By Van E. Wood

1. Introduction. The integrals

$$I_n^k(t) = (2\pi i)^{-1} \int_{-\infty}^{(0+)} e^{zt} z^{n-1} (\ln z)^k \, dz, \qquad \operatorname{Re} t > 0,$$

occur in the asymptotic expansions of the solutions of heat conduction problems in regions bounded internally by a circular cylinder [1], in problems on the flow of fluids through porous media [2], in electron slowing-down problems [3], and elsewhere. It should be recognized that these integrals are *not* in general the inverse Laplace transforms of $z^{n-1}(\ln z)^k$, since the contour does not surround the singularity occurring at $z = 1$ when $k < 0$. We will consider only cases where t is real and n and k are integers. For k nonnegative, the integrals can be expressed in terms of polygamma functions [2]. For nonnegative n and negative k, they can be expressed, by means of change of variables and integrations by parts, in terms of derivatives of Ramanujan's integral [4],

$$I_R(t) = \int_0^\infty e^{-tx} x^{-1} (\pi^2 + \ln^2 x)^{-1} \, dx.$$

This function is in turn related to the ν-functions of Volterra and others [5, 6], which are useful in the solution of certain integral equations. In this paper, we discuss properties and numerical values of Ramanujan's integral, its derivatives, and the related contour integrals.

2. Relation to Other Integrals. Using the recurrence relations

$$dI_n^k(t)/dt = I_{n+1}^k(t), \tag{1a}$$

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