Practical $L^p$ Polynomial Approximation

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Let $p$ denote a positive integer, and let $P_{pq}(x)$ denote the polynomial of degree $q$ with coefficient of $x^q$ unity which gives the least value when the $p$th power of its modulus is integrated over $(-1, 1)$. The extension of what is given below to an interval other than $(-1, 1)$ is trivial. The existence and uniqueness of $P_{pq}(x)$ follows from the general theory of $L^p$ approximation, see, for example, [1].

If we write $P_{pq}(x) = (x - x_1) \cdots (x - x_q)$, we may deduce from elementary considerations that the $x_r$ are real, interior to $(-1, 1)$, symmetrically placed with respect to $x = 0$ and distinct.

Clearly $P_{1q} = 2^{-q} U_q(x)$, where $U_q(x)$ is the Chebyshev polynomial of the second kind of degree $q$, and $P_{2q}(x) = [2^2(q!)^2/(2q)!] P_q(x)$, where $P_q(x)$ is the Legendre polynomial of degree $q$. Since $U_q(x)$ and $P_q(x)$ are the ultraspherical polynomials $P_q^{(1)}(x)$ and $P_q^{(1/2)}(x)$ respectively, it may be conjectured that $P_{pq}(x) = k_q P_q^{(1/2)}(x)$; however this is disproved by the fact that the zeros of $P_{4q}(x)$ are about $+0.629$, while those of $P_2^{(1/2)}(x)$ are about $+0.632$.

Below we tabulate the positive zeros of $P_{pq}(x)$ to five decimal places for $p, q = 2(1)7$ (table 1). To each positive zero there corresponds a negative zero of equal magnitude, and, for odd $q$, $x = 0$ is also a zero. The coefficients of $P_{pq}(x)$ may be obtained from the given zeros. We also tabulate $L^p(P_{pq})$, where $L^p(f)$ denotes $\{\int_{-1}^{1} |f(x)|^p dx\}^{1/p}$. The case $p = 1$ was not included since we have $P_{1q}(x) = (x - \cos \left(\pi/(q + 1)\right)) \cdots (x - \cos \left(q\pi/(q + 1)\right))$ and $L^1(P_{1q}) = 2^{-q}$. On the other hand the case $p = 2$ was included for convenience and purposes of comparison, although $P_{2q}(x)$ is essentially a well-known polynomial. Note that, for all $p$, $P_{p0}(x) = 1$ and $P_{p1}(x) = x$.

The zeros $x_r$ were evaluated as follows. Suppose first that $q$ is even with $Q = \lfloor q/2 \rfloor$, then we may take $P_{pq}(x) = (x^2 - x_1^2) \cdots (x^2 - x_q^2)$ where $0 < x_1 < \cdots < x_q < 1$. In particular, for $p = 1$ we have $x_r = \cos \left((Q - r + 1)\pi/(q + 1)\right)$. Since $\int_{-1}^{1} |P_{pq}(x)|^p dx$ is a minimum we obtain

$$\int_{0}^{1} \{ |x^2 - x_1^2| \cdots |x^2 - x_q^2|/(x^2 - x_r^2) \} dx = 0.$$

We now solve these equations iteratively by taking $x_{r0}(p) = x_r(p - 1)$ (where the notation shows the dependence on $p$) and $x_{r+1}^2 = x_{r_s}^2 + \epsilon_{rs}$ where $pbr_s \epsilon_{rs} + \cdots + pb_{rs} \epsilon_{qs} - b_{rs} \epsilon_{rs} = a_{rs}$,

$$a_{rs} = \int_{0}^{1} \{ |x^2 - x_{1s}| \cdots |x^2 - x_{qs}|/(x^2 - x_{rs}^2) \} dx$$

and

$$b_{rs} = \int_{0}^{1} \{ |x^2 - x_{1s}| \cdots |x^2 - x_{qs}|/(x^2 - x_{rs}^2)(x^2 - x_{is}^2) \} dx.$$
The tables enable practical $L^p$ polynomial approximation to be performed approximately in a variety of ways. Thus, to obtain an $L^p$ polynomial approximation of degree $q - 1$ over $(-1, 1)$ to a function $f(x)$, we may collocate at the zeros of $P_{pq}(x)$, and so get an error of $(x - x_1) \cdots (x - x_q)f^{(q)}(\xi)/q! = P_{pq}(x)f^{(q)}(\xi)/q!$, where $\xi$ belongs to $(-1, 1)$, assuming the existence and continuity of $f^{(q)}(x)$ over the interval. In the $L^p$ sense this error is the least possible if $f^{(q)}(x)$ is constant over the interval, and in other cases the method may be assumed to give a good approxima-
tion to the desired polynomial.* Alternatively, if \( f(x) \) possesses a power series expansion we may truncate the latter at the term involving \( x^Q \), where \( Q \geq q \), and rearrange the resulting polynomial in terms of \( P_{p0}(x), P_{p1}(x), \ldots, P_{pq}(x) \) and "economize" (in an analogous way to Lanczos' economization procedure [2]) to a polynomial of degree \( q - 1 \) expressed in terms of \( P_{p0}(x), P_{p1}(x), \ldots, P_{p(q-1)}(x) \).

In this case an upper bound for the error in the \( L^p \) sense is given by the sum of the \( p \)th root of the integrals of the \( p \)th power of the modulus of the neglected terms. In both methods the \( L^p \) error may be estimated by using the tabulated values of \( L^p(P_{pq}) \). In special cases it may be convenient to use other methods to approximate \( f(x) \). Thus, if \( f(x) \) satisfies a suitable differential equation, we could use a procedure rather like that of Clenshaw [3] for Chebyshev-type approximation. A similar technique may be used if \( f(x) \) is a rational function.

Finally we consider a simple numerical example in which \( f(x) = \log \left( \frac{1}{4} + x/2 \right) \), \( p = 3 \) and \( q = 3 \). Starting with the approximation \( f(x) = \log \left( \frac{1}{4} + x/3 \right) - x^2/18 + x^3/81 \) a simple iterative method gives the best approximation as \( f(x) = 0.40579 + 0.33304x - 0.05852x^2 + 0.01346x^3 \), with an \( L^3 \) error of about 0.00037. The results given by using the tables in conjunction with the principal approximate methods outlined above are as follows.

Collocation: \( f(x) = 0.40578 + 0.33325x - 0.05850x^2 + 0.01313x^3 \), \( L^3 \) error about 0.00039.

Economization with \( Q = 5 \): \( f(x) = 0.40576 + 0.33312x - 0.05833x^2 + 0.01328x^3 \), \( L^3 \) error about 0.00039.

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* If \( a(x) \) is an approximation to a function \( f(x) \), we call \( L^p(a - f) \) the error in the \( L^p \) sense, or simply the \( L^p \) error.