of order $m$, depends only on $\lambda$ and, by (4.6), its $(\alpha, \beta)$ element is a function of $\alpha - \beta$. If we define $N = (e_2, \cdots, e_m, 0)$, where $I = (e_1, \cdots, e_m)$, then

$$L^i = \sum_{i=0}^{m-1} \frac{p_i^{(s)}(\lambda)}{v} p_i(\lambda) N_i,$$

and is a polynomial in $N_i$. Since $J_i$ is also a polynomial in $N_i$ it must commute with $L_i$.

The above results were derived for $H \in UHM$. However, properties (ii) and (iii) generalize immediately to all Hessenberg matrices by the remarks at the beginning of Section 2.

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2. V. N. Faddeeva, Computational Methods of Linear Algebra, Dover, New York, 1959, p. 20.

An Elimination Method for Computing the Generalized Inverse*

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0. Notations. We denote by

$A$ an $m \times n$ complex matrix,

$A^*$ the conjugate transpose of $A$,

$A_{j}$, $j = 1, \cdots, n$ the $j$th column of $A$,

$A^+$ the generalized inverse of $A$ [7],

$H$ the Hermite normal form of $A$, [6, pp. 34–36],

$Q^{-1}$ the nonsingular matrix satisfying

$$H = Q^{-1}A,$$

$e_i$, $i = 1, \cdots, m$ the $i$th unit vector $e_i = (\delta_{ij})$,

$r$ the rank of $A$ ($= \text{rank } H$).

1. Method. The Hermite normal form of $A$ is written as

$$H = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

where $B$ is $r \times n$. 

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Combining (1) and (2) we have:

\[(3) \quad A = QH = [P, R] \begin{bmatrix} B \\ 0 \end{bmatrix} = PB,\]

where \([P, R]\) is the corresponding partition of \(Q\). Having displayed the \(m \times n\) matrix \(A\) of rank \(r\) as a product of the \(m \times r\) matrix \(P\) and the \(r \times n\) matrix \(B\), which are both of rank \(r\), we have as in [4]

\[(4) \quad A^+ = B^+P^+ = B^*(BB^*)^{-1}(P^*P)^{-1}P^*\]

therefore

\[(5) \quad A^+ = B^*(P^*PBB^*)^{-1}P^*\]
and by (3)

\[(6) \quad A^+ = B^*(P^*AB^*)^{-1}P^*.\]

The method can be summarized as follows:

*Step 1.* Given \(A\) obtain \(H\) by Gaussian elimination.

*Step 2.* From \(H\) determine \(P\) as follows:

The \(i\)th column of \(P\), \(P_i\), \(i = 1, \ldots, r\) is

\[(7) \quad P_i = A_j \quad \text{if} \quad H_j = e_i, \quad j = 1, \ldots, n.\]

*Step 3.* Calculate \(P^*AB^*\).

*Step 4.* Invert \(P^*AB^*\).

*Step 5.* Calculate \(A^+\) using (6).

2. **Remarks.**

(i) From (7) we conclude that in order to obtain \(P\) it is unnecessary

to keep track of the elementary operations involved in finding \(H\), e.g. [5].

(ii) Representation (4), as a computational method, was suggested by Gre-ville [4], Householder [5] and Frame [2]. The novelty of the present paper lies in

equation (6) and Step 2 above.

(iii) Like other elimination methods for computing \(A^+\), e.g. [1], the method

proposed here depends critically on the correct determination of rank \(A\), e.g. the

discussion in [3].

(iv) The advantage of method (6) over the elimination method of [1] is that

here the matrix \(A^*A\) (or \(AA^*\)) is not computed. However, other matrix multipli-
cations are involved in this method.

3. **Example.** For

\[A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}\]

we obtain by Gaussian elimination

\[
\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = H
\]

since \(H_2 = e_1\) we have \(P_1 = A_2\), and since \(H_3 = e_2\) we have \(P_2 = A_3\). Hence for
A = PB we have
\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 2 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
1 & 1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 & -1
\end{bmatrix}
\]

and
\[
P^*AB^* = \begin{bmatrix}
12 & -3 \\
5 & 0
\end{bmatrix}
\]

from which
\[
(P^*AB^*)^{-1} = \frac{1}{15} \begin{bmatrix}
0 & 3 \\
-5 & 12
\end{bmatrix}
\]

hence
\[
A^+ = B^*(P^*AB^*)^{-1}P^* = \frac{1}{15} \begin{bmatrix}
0 & 0 & 0 \\
0 & 3 & 3 \\
-5 & 7 & 2 \\
5 & -4 & 1 \\
5 & -4 & 1
\end{bmatrix}
\]