of order m_i, depends only on λ_i and, by (4.6), its (α, β) element is a function of $\alpha - \beta$. If we define $N_i = (e_2, \cdots, e_{m_i}, 0)$, where $I = (e_1, \cdots, e_{m_i})$, then

$$L^{ii} = \sum_{\nu=0}^{m_i-1} \frac{p_i^{(\nu)}(\lambda_i)}{\nu!\, p_i(\lambda_i)} N_i^{\nu}, \tag{4.8}$$

and is a polynomial in N_i. Since J_i is also a polynomial in N_i it must commute with L^{ii}.

The above results were derived for $H \in$ UHM. However, properties (ii) and (iii) generalize immediately to all Hessenberg matrices by the remarks at the beginning of Section 2.

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An Elimination Method for Computing the Generalized Inverse*

By Leopold B. Willner

0. Notations. We denote by

A an $m \times n$ complex matrix,
A^* the conjugate transpose of A,
A_j, $j = 1, \cdots, n$ the jth column of A,
A^+ the generalized inverse of A [7],
H the Hermite normal form of A, [6, pp. 34–36],
Q^{-1} the nonsingular matrix satisfying

$$H = Q^{-1}A, \tag{1}$$

e_i, $i = 1, \cdots m$ the ith unit vector $e_i = (\delta_{ij})$,
r the rank of A (= rank H).

1. Method. The Hermite normal form of A is written as

$$H = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \text{where } B \text{ is } r \times n. \tag{2}$$

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Combining (1) and (2) we have:

$$A = QH = [P, R] \begin{bmatrix} B \\ 0 \end{bmatrix} = PB, \tag{3}$$

where $[P, R]$ is the corresponding partition of Q. Having displayed the $m \times n$ matrix A of rank r as a product of the $m \times r$ matrix P and the $r \times n$ matrix B, which are both of rank r, we have as in [4]

$$A^+ = B^+P^+ = B^*(BB^*)^{-1}(P^*P)^{-1}P^* \tag{4}$$

therefore

$$A^+ = B^*(P^*PBB^*)^{-1}P^* \tag{5}$$

and by (3)

$$A^+ = B^*(P^*AB^*)^{-1}P^*. \tag{6}$$

The method can be summarized as follows:

Step 1. Given A obtain H by Gaussian elimination.

Step 2. From H determine P as follows:

The ith column of P, P_i, $i = 1, \cdots, r$ is

$$P_i = A_j \quad \text{if} \quad H_j = e_i, \qquad j = 1, \cdots, n. \tag{7}$$

Step 3. Calculate P^*AB^*.

Step 4. Invert P^*AB^*.

Step 5. Calculate A^+ using (6).

2. Remarks. (i) From (7) we conclude that in order to obtain P it is unnecessary to keep track of the elementary operations involved in finding H, e.g. [5].

(ii) Representation (4), as a computational method, was suggested by Greville [4], Householder [5] and Frame [2]. The novelty of the present paper lies in equation (6) and Step 2 above.

(iii) Like other elimination methods for computing A^+, e.g. [1], the method proposed here depends critically on the correct determination of rank A, e.g. the discussion in [3].

(iv) The advantage of method (6) over the elimination method of [1] is that here the matrix A^*A (or AA^*) is not computed. However, other matrix multiplications are involved in this method.

3. Example. For

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}$$

we obtain by Gaussian elimination

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = H$$

since $H_2 = e_1$ we have $P_1 = A_2$, and since $H_3 = e_2$ we have $P_2 = A_3$. Hence for

$A = PB$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

and

$$P^*AB^* = \begin{bmatrix} 12 & -3 \\ 5 & 0 \end{bmatrix}$$

from which

$$(P^*AB^*)^{-1} = \frac{1}{15}\begin{bmatrix} 0 & 3 \\ -5 & 12 \end{bmatrix}$$

hence

$$A^+ = B^*(P^*AB^*)^{-1}P^* = \frac{1}{15}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ -5 & 7 & 2 \\ 5 & -4 & 1 \\ 5 & -4 & 1 \end{bmatrix}.$$

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