Eigenvalues and Eigenvectors of Hilbert Matrices of Order 3 Through 10

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Elements of the Hilbert matrix of order n are defined by

$$(1) \qquad H_{ij} = 1/(i+j-1), \qquad i = 1, \cdots, n; \qquad j = 1, \cdots, n.$$

Because of its relatively low condition number [4], it is used frequently as a test matrix. Eight-place values of the dominant eigenvalue and the elements of the corresponding eigenvector were first published by Fairthorne and Miller [1] for matrices of order 2, 3, 4, 5, 6, 8, 10 and 20. More recently Denman and Ettinger [2] gave 17-place values of the greatest and least eigenvalues together with the elements of corresponding eigenvectors for matrices of orders 2, 4, 6, 8, 10.

In the present paper Denman and Ettinger's results are extended to include all of the eigenvalues and eigenvectors for matrices of order 3 through 10. These were computed by the cyclic Jacobi method [5]* from both the original matrix and the inverse. The computations were made with 25-digit arithmetic and the final results rounded to sixteen figures. In all cases both sets of results gave agreement to sixteen figures, with even better agreement in the lower orders. This is to be considered as a reliable check, since the elements of the inverse matrix can be expressed as exact numbers [3]. In addition, the correctness of the smallest eigenvalue can be verified by multiplying the inverse matrix by the corresponding eigenvector, and since both the rows of the inverse matrix and the elements of the eigenvector have alternating signs, there can be no loss of significance due to subtraction. A further check is obtained by comparing the product of the eigenvalues with the determinant whose value is known explicitly [3]. Although the Jacobi method produces eigenvectors which are orthonormal, the final results presented here have been renormalized so as to make the largest element unity in order to facilitate comparison with the results of [2].

In general the agreement between the present results and those of [2] is good, with the exception of the smallest eigenvalue of H_{10}, which is given in the latter paper as

$$1.09315\ 38198\ 57659\ 9 \times 10^{-13}$$

as compared to the value

$$1.09315\ 38193\ 79665\ 8 \times 10^{-13}$$

obtained here. However, since there is complete agreement in the elements of the corresponding eigenvectors, it appears that the above discrepancy is due to a misprint rather than an actual computational error. Other minor discrepancies are noted below:

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* The use of this method was motivated by the fact that it was already available as a subroutine for the IBM 1620 on which all of the computations were made.

Dominant eigenvalue of H_4: last digit = 2,
Smallest eigenvalue of H_4: last two digits = 55,
Dominant eigenvalue of H_8: last digit = 5.

In the tables which follow, all of the results are given in floating-point form: $a \times 10^P$.

Order of Matrix = 3

Eigenvalues			P	*Eigenvectors*			P
1.40831	89271	23654	−00	1.00000	00000	00000	−00
				5.56032	55563	05693	−01
				3.90907	94792	51080	−01
1.22327	06585	39058	−01	−8.43517	43276	29785	−01
				8.13998	17376	62614	−01
				1.00000	00000	00000	−00
2.68734	03557	73529	−03	−1.78857	98493	23438	−01
				1.00000	00000	00000	−00
				−9.64868	00204	55515	−01

Order of Matrix = 4

Eigenvalues			P	*Eigenvectors*			P
1.50021	42800	59243	−00	1.00000	00000	00000	−00
				5.70172	08366	32358	−01
				4.06778	98802	75292	−01
				3.18140	96887	37940	−01
1.69141	22022	14500	−01	+1.00000	00000	00000	−00
				−6.36518	90190	07507	−01
				−8.75450	79607	67703	−01
				−8.83129	58721	03381	−01
6.73827	36057	60748	−03	−2.41517	71638	15848	−01
				1.00000	00000	00000	−00
				−1.35093	31925	07654	−01
				−8.60314	35862	04442	−01
9.67023	04022	58689	−05	+3.68876	82614	14105	−02
				−4.15349	28778	03112	−01
				1.00000	00000	00000	−00
				−6.50171	21973	36798	−01

Order of Matrix = 5

Eigenvalues			P	*Eigenvectors*			P
1.56705	06910	98231	−00	1.00000	00000	00000	−00
				5.80566	92249	80478	−01
				4.18800	95256	90560	−01
				3.30061	05409	17674	−01
				2.73258	24401	62320	−01

Order of Matrix = 5 (cont.)

Eigenvalues			P	*Eigenvectors*			P
2.08534	21861	10133	−01	+1.00000	00000	00000	−00
				−4.58425	80576	61740	−01
				−7.05925	82907	15063	−01
				−7.37537	92074	31147	−01
				−7.12798	94314	80946	−01
1.14074	91623	41981	−02	−2.95833	43954	91379	−01
				1.00000	00000	00000	−00
				1.66348	46563	67509	−01
				−4.27528	04665	91248	−01
				−7.80543	77407	62442	−01
3.05898	04015	11917	−04	+7.06702	26210	87525	−02
				−6.48336	02593	66261	−01
				1.00000	00000	00000	−00
				3.49178	63233	06241	−01
				−8.35542	93387	42830	−01
3.28792	87721	71863	−06	−8.04735	96573	69526	−03
				1.52103	86654	52718	−01
				−6.59762	08136	21921	−01
				1.00000	00000	00000	−00
				−4.90419	53143	50719	−01

Order of Matrix = 6

Eigenvalues			P	*Eigenvectors*			P
1.61889	98589	24339	−00	1.00000	00000	00000	−00
				5.88628	54342	55432	−01
				4.28327	28442	89561	−01
				3.39661	89183	87095	−01
				2.82523	58794	21492	−01
				2.42337	81112	28495	−01
2.42360	87057	52096	−01	+1.00000	00000	00000	−00
				−3.43477	76103	67806	−01
				−5.95389	61269	85598	−01
				−6.42274	99431	02546	−01
				−6.31671	46805	61395	−01
				−6.03204	01490	85321	−01
1.63215	21319	87582	−02	−3.44477	74040	00321	−01
				1.00000	00000	00000	−00
				3.31669	05639	78445	−01
				−1.90443	48397	72404	−01
				−5.19908	55937	27446	−01
				−7.20650	57788	73129	−01

Order of Matrix = 6 (cont.)

Eigenvalues			*P*	*Eigenvectors*			*P*
6.15748	35418	26577	−04	−1.15089	16226	58221	−01
				9.07815	69634	66591	−01
				−9.90373	92462	04362	−01
				−7.71318	49997	79162	−01
				8.69902	39991	00457	−02
				1.00000	00000	00000	−00
1.25707	57122	62519	−05	1.84443	82298	42188	−02
				−2.97466	27961	49800	−01
				1.00000	00000	00000	−00
				−7.34137	29699	37382	−01
				−7.30764	18529	36246	−01
				7.59856	90405	64665	−01
1.08279	94845	65550	−07	+1.80948	25414	40515	−03
				−5.16182	53594	24858	−02
				+3.48907	75253	55039	−01
				−9.06717	68457	84127	−01
				1.00000	00000	00000	−00
				−3.93741	11149	37020	−01

Order of Matrix = 7

Eigenvalues			*P*	*Eigenvectors*			*P*
1.66088	53389	26931	−00	1.00000	00000	00000	−00
				5.95122	63106	51334	−01
				4.36126	49735	70395	−01
				3.47622	28057	85343	−01
				2.90284	56422	45982	−01
				2.49777	30606	63215	−01
				2.19495	43192	32110	−01
2.71920	19814	93452	−01	+1.00000	00000	00000	−00
				−2.61651	87231	55985	−01
				−5.15876	57109	07207	−01
				−5.73403	92060	79247	−01
				−5.72924	00712	20064	−01
				−5.52870	16536	01293	−01
				−5.26499	39668	34787	−01
2.12897	54908	32795	−02	−3.88993	12692	28422	−01
				1.00000	00000	00000	−00
				4.40423	18707	70565	−01
				−3.43693	09768	67312	−02
				−3.48496	99421	85792	−01
				−5.48611	13060	83639	−01
				−6.74584	79000	81032	−01

Order of Matrix = 7 (cont.)

Eigenvalues			*P*	*Eigenvectors*			*P*
1.00858	76107	70142	−03	−1.42730	60664	90882	−01
				1.00000	00000	00000	−00
				−8.08037	82159	87594	−01
				−8.76415	71705	77919	−01
				−3.24986	39912	94217	−01
				3.46758	27944	15234	−01
				9.67685	27364	04640	−01
2.93863	68145	92969	−05	2.54375	48028	50871	−02
				−3.62466	87695	55796	−01
				1.00000	00000	00000	−00
				−3.18671	25647	34205	−01
				−7.90476	81764	75158	−01
				−2.94027	65031	95887	−01
				7.64617	63467	28869	−01
4.85676	33615	74250	−07	−3.82934	35999	28926	−03
				9.58555	22029	61430	−02
				−5.40843	92603	82367	−01
				1.00000	00000	00000	−00
				−2.70501	09557	80551	−01
				−8.43244	13447	80657	−01
				5.65766	58320	00075	−01
3.49389	86059	91218	−09	3.59098	91821	95847	−04
				−1.44149	72273	50558	−02
				1.39474	73803	77168	−01
				−5.44035	08875	84887	−01
				1.00000	00000	00000	−00
				−8.65947	69018	12042	−01
				2.84831	36565	59360	−01

Order of Matrix = 8

Eigenvalues			*P*	*Eigenvectors*			*P*
1.69593	89969	21949	−00	1.00000	00000	00000	−00
				6.00504	24575	79538	−01
				4.42671	55401	19186	−01
				3.54370	44699	96978	−01
				2.96918	57844	45071	−01
				2.56180	92948	69805	−01
				2.25629	36880	82276	−01
				2.01790	18703	79183	−01
2.98125	21131	69307	−01	+1.00000	00000	00000	−00
				−1.99641	07668	62729	−01
				−4.55006	08109	55082	−01
				−5.20378	81437	80070	−01
				−5.27565	95376	14758	−01
				−5.13976	52248	24774	−01
				−4.92889	11462	14828	−01
				−4.69617	42309	87346	−01

Order of Matrix = 8 (cont.)

Eigenvalues			*P*	*Eigenvectors*			*P*
2.62128	43578	11905	−02	−4.30353	53605	31482	−01
				1.00000	00000	00000	−00
				5.19670	85157	87076	−01
				7.95547	21960	88871	−02
				−2.23317	97982	52556	−01
				−4.23090	40968	60939	−01
				−5.53618	29278	79063	−01
				−6.38180	47813	30543	−01
1.46768	81177	41867	−03	−1.57264	17782	81082	−01
				1.00000	00000	00000	−00
				−6.11217	36807	18912	−01
				−8.34786	09555	82616	−01
				−5.01055	24897	20307	−01
				−2.04888	90803	70457	−02
				4.52548	32316	57058	−01
				8.67560	91295	41801	−01
5.43694	33697	49942	−05	3.30983	85076	81348	−02
				−4.26438	76153	95457	−01
				1.00000	00000	00000	−00
				−5.12022	50480	36401	−02
				−6.72019	76374	56363	−01
				−5.85721	70581	51348	−01
				−2.88541	78933	63593	−02
				7.65890	29707	01414	−01
1.29433	20918	72811	−06	−6.57331	00511	27815	−03
				1.47989	76372	18060	−01
				−7.22416	37481	59994	−01
				1.00000	00000	00000	−00
				2.12540	06545	33205	−01
				−7.21939	02421	98936	−01
				−6.07829	90639	21476	−01
				7.04255	03469	70941	−01
1.79887	37458	17577	−08	−9.20944	89718	87311	−04
				3.30647	67571	34515	−02
				−2.77763	32839	45767	−01
				8.69385	31714	42472	−01
				−9.95882	21590	46623	−01
				−1.31307	33940	75290	−01
				1.00000	00000	00000	−00
				−4.97329	08134	30048	−01
1.11153	89663	72442	−10	−6.86103	92145	12811	−05
				3.68787	70518	27661	−03
				−4.82672	54524	49843	−02
				2.61713	39967	61041	−01
				−7.05747	34717	96188	−01
				1.00000	00000	00000	−00
				−7.12509	13818	01248	−01
				2.01241	83438	37764	−01

Order of Matrix = 9

Eigenvalues			*P*	*Eigenvectors*			*P*
1.72588	26609	01847	−00	1.00000	00000	00000	−00
				6.05062	73643	51117	−01
				4.48271	58106	99431	−01
				3.60192	03013	69706	−01
				3.02681	26027	11120	−01
				2.61776	14674	05719	−01
				2.31016	02171	39396	−01
				2.06956	70822	18468	−01
				1.87577	43844	56729	−01
3.21633	12229	92068	−01	+1.00000	00000	00000	−00
				−1.50563	00038	85823	−01
				−4.06370	51543	62231	−01
				−4.77760	59330	52471	−01
				−4.90983	21999	98590	−01
				−4.82553	42744	47961	−01
				−4.65723	08579	25155	−01
				−4.45945	07919	47498	−01
				−4.25620	54046	16327	−01
3.10389	25781	26833	−02	−4.69215	60414	24556	−01
				1.00000	00000	00000	−00
				5.81298	48942	56370	−01
				1.68338	31445	98111	−01
				−1.25629	34840	26744	−01
				−3.25116	29771	00839	−01
				−4.59281	69753	28175	−01
				−5.49127	85754	13251	−01
				−6.08710	62023	19202	−01
1.97893	38602	15924	−03	−1.70653	85569	90212	−01
				1.00000	00000	00000	−00
				−4.64717	14467	22524	−01
				−7.75951	09562	54300	−01
				−5.78327	97815	76324	−01
				−2.21347	44112	47116	−01
				1.53939	04201	98640	−01
				4.96040	77413	22528	−01
				7.89930	71224	23032	−01
8.75808	50514	59757	−05	4.13830	29491	86680	−02
				−4.90024	66072	79400	−01
				1.00000	00000	00000	−00
				1.41406	98631	98621	−01
				−5.29978	89637	04739	−01
				−6.54975	57014	80830	−01
				−3.71913	95311	21723	−01
				1.44275	27836	59178	−01
				7.66821	37097	76299	−01

Order of Matrix = 9 (cont.)

Eigenvalues			*P*	*Eigenvectors*			*P*
2.67301	34105	99414	−06	−1.05472	67848	64242	−02
				2.17325	36475	07308	−01
				−9.40220	91065	07029	−01
				1.00000	00000	00000	−00
				5.70068	66549	30026	−01
				−4.36808	34364	41685	−01
				−8.57092	52955	24884	−01
				−3.92465	30179	95107	−01
				8.60212	23900	26319	−01
5.38561	33485	22494	−08	−1.57026	36778	04265	−03
				5.13643	34130	24127	−02
				−3.83966	02990	96905	−01
				1.00000	00000	00000	−00
				−7.07732	14392	12621	−01
				−6.33233	77263	71795	−01
				4.61853	32026	77920	−01
				8.23643	49129	50087	−01
				−6.11750	86298	36100	−01
6.46090	54226	38582	−10	+1.86895	58009	44782	−04
				−9.11142	60410	63340	−03
				1.06097	02731	82322	−01
				−4.88842	63971	64796	−01
				1.00000	00000	00000	−00
				−7.08369	80999	49995	−01
				−4.70608	86461	34285	−01
				9.44390	21369	08030	−01
				−3.73891	52081	99850	−01
3.49967	64029	11493	−12	1.36620	49070	13275	−05
				−9.47535	57566	95441	−04
				1.61058	61751	07205	−02
				−1.15383	19398	86663	−01
				4.24473	99502	14224	−01
				−8.68875	53580	11983	−01
				1.00000	00000	00000	−00
				−6.05131	38110	13442	−01
				1.49754	18355	81289	−01

Order of Matrix = 10

Eigenvalues			*P*	*Eigenvectors*			*P*
1.75191	96702	65178	−00	1.00000	00000	00000	−00
				6.08991	91436	96503	−01
				4.53138	29895	94215	−01
				3.65286	01340	21510	−01
				3.07753	04744	55016	−01
				2.66725	18429	30508	−01
				2.35801	30798	24843	−01
				2.11563	96395	15401	−01
				1.92005	12818	61191	−01
				1.75860	03439	31029	−01

Order of Matrix = 10 (*cont.*)

Eigenvalues			*P*	*Eigenvectors*			*P*
3.42929	54848	35091	−01	1.00000	00000	00000	−00
				−1.10465	17177	43785	−01
				−3.66282	37964	87492	−01
				−4.42425	91767	28277	−01
				−4.60536	94518	17078	−01
				−4.56341	25730	07357	−01
				−4.43036	22451	70023	−01
				−4.26171	31611	63669	−01
				−4.08260	80801	73888	−01
				−3.90483	78675	18817	−01
3.57418	16271	63924	−02	−5.06044	64866	39978	−01
				1.00000	00000	00000	−00
				6.31415	38757	61648	−01
				2.40699	24192	57739	−01
				−4.58618	75111	07758	−02
				−2.45040	59873	37491	−01
				−3.82178	19752	80346	−01
				−4.76401	86973	73599	−01
				−5.40836	48066	89185	−01
				−5.84363	76347	92411	−01
2.53089	07686	70038	−03	−1.83100	74913	05559	−01
				1.00000	00000	00000	−00
				−3.50413	96416	78791	−01
				−7.15015	97273	02427	−01
				−6.09646	75122	58232	−01
				−3.39133	24904	62540	−01
				−3.36629	40180	83149	−02
				2.55456	55324	15149	−01
				5.10538	49054	81802	−01
				7.27979	70808	98943	−01
1.28749	61427	63771	−04	5.02691	93708	19130	−02
				−5.53715	19652	43531	−01
				1.00000	00000	00000	−00
				2.90771	16075	67017	−01
				−3.93565	04576	91000	−01
				−6.42818	76489	72120	−01
				−5.30534	27508	17633	−01
				−1.92318	51809	90450	−01
				2.64461	64931	15005	−01
				7.68640	20927	86739	−01
4.72968	92931	82348	−06	1.34197	07196	31349	−02
				−2.56349	00053	68558	−01
				1.00000	00000	00000	−00
				−8.26254	49309	85651	−01
				−7.30398	87720	04190	−01
				9.79799	70771	83957	−02
				6.90035	19449	49648	−01
				7.09559	50102	98084	−01
				1.52328	09663	07994	−01
				−8.64618	11259	32675	−01

Order of Matrix = 10 (*cont.*)

Eigenvalues			*P*	*Eigenvectors*			*P*
1.22896	77387	51175	−07	−2.20211	62655	20837	−03
				6.65308	34126	35937	−02
				−4.50423	02451	95166	−01
				1.00000	00000	00000	−00
				−3.97406	18263	63392	−01
				−7.52617	34673	95069	−01
				−5.19153	37204	50187	−02
				6.56558	24094	19764	−01
				5.76366	49789	61755	−01
				−6.46988	57273	97804	−01
2.14743	88173	50479	−09	3.29136	10483	09510	−04
				−1.47645	74515	12742	−02
				1.55750	54872	45585	−01
				−6.25397	69321	29557	−01
				1.00000	00000	00000	−00
				−2.34914	01869	53356	−01
				−7.83112	53616	02477	−01
				1.07843	33964	72780	−01
				8.66944	29913	73357	−01
				−4.72962	81021	99039	−01
2.26674	67477	62926	−11	−3.73254	77290	76785	−05
				2.37327	10245	65077	−03
				−3.64962	96430	35628	−02
				2.29661	97587	73360	−01
				−6.96603	89892	21241	−01
				1.00000	00000	00000	−00
				−3.68315	31552	82386	−01
				−6.99899	14802	67557	−01
				8.48690	27725	93223	−01
				−2.79403	11288	36095	−01
1.09315	38193	79666	−13	2.71471	31336	04098	−06
				−2.36061	26295	90383	−04
				5.05289	73867	16890	−03
				−4.61160	40049	98925	−02
				2.20661	51772	89104	−01
				−6.08176	78395	43368	−01
				1.00000	00000	00000	−00
				−9.68158	87951	22191	−01
				5.09073	58516	71383	−01
				−1.12104	94021	47474	−01

Values of the Determinant Δ_n of the Hilbert Matrix of Order n

n	Δ_n					*P*
2	8.33333	33333	33333	33333	33333	−2
3	4.62962	96296	29629	62962	96296	−4
4	1.65343	91534	39153	43915	34392	−7
5	3.74929	51325	15087	16361	32407	−12
6	5.36729	98873	58687	73278	88304	−18
7	4.83580	26239	26116	93211	98556	−25
8	2.73705	01137	91513	01664	20433	−33
9	9.72023	43119	24999	86288	94723	−43
10	2.16417	92264	31491	86906	05950	−53

*Products of Computed Eigenvalues of the Hilbert Matrix of Order n**

$$\left[\prod_{i=1}^{n} \lambda_i\right]$$

n						P
3	4.62962	96296	29629	62962	963	−4
4	1.65343	91534	39153	43915	344	−7
5	3.74929	51325	15087	16361	315	−12
6	5.36729	98873	58687	73278	556	−18
7	4.83580	26239	26116	93207	521	−25
8	2.73705	01137	91513	01622	963	−33
9	9.72023	43119	24999	83089	307	−43
10	2.16417	92264	31492	19924	028	−53

* Unrounded values were used in computing the product.

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