A Steady State Phase Change Problem*

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In a previous paper [4] the solution to the Stefan problem for a one-dimensional semi-infinite slab with constant boundary and initial conditions was shown to be given by the limit of solutions to a nonlinear parabolic equation for the "specific internal energy." In this paper we obtain the same result for the Stefan problem in a bounded two- or three-dimensional domain, with constant boundary conditions. This result further justifies the application of the methods of [3], [4] to the Stefan problem in higher dimensions.

In Section 1 the problem to be solved is stated, and a simple solution given. In Sections 2, 3 this solution is shown to be obtainable from a limit of solutions to a related problem for the specific internal energy, as well as a solution to a related problem in the calculus of variations.

1. Notation and Statement of the Problem. Let $\mathcal{L}$ be a bounded region of the x, y plane having a smooth boundary Γ and consisting of material which undergoes a change of phase, from Phase "I" to Phase "II," at the critical temperature T_c (see Fig. 1); our results apply as well for a three-dimensional region. (Phases I and II can represent "frozen" and "melted" states of the material.) Let H be the latent heat of the material which is lost in the transition from Phase II to Phase I, c_1, K_1 and c_2, K_2 the specific heat and conductivity of Phase I and Phase II material, respectively, and $\kappa_i = K_i/c_i\rho$, $i = 1, 2$, where ρ is the density of Phase I and II material, which we assume to be the same.

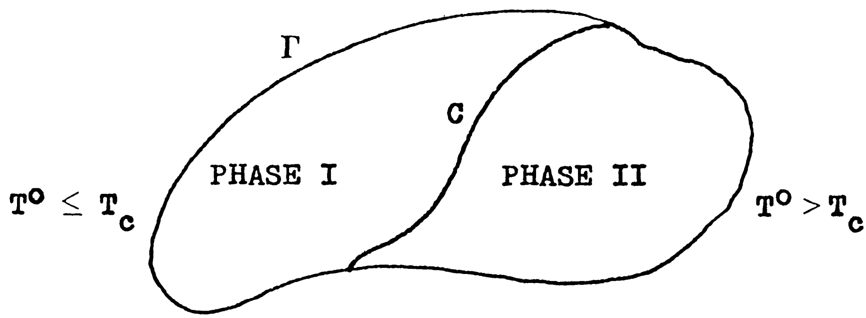

FIGURE 1. The domain $\mathcal{L}$

Suppose that the temperature T of the boundary Γ is given by the function

$$T(\sigma) = T^0(\sigma) \tag{1a}$$

(with σ the arc length on Γ) and maintained at this temperature for all time. T^0 is to be a bounded and piecewise continuous function of σ assuming values above and below T_c. Then the steady state temperature $T(x, y)$ at points (x, y) of $\mathcal{L}$ is har-

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monic throughout $\mathcal{L}$ except across certain curves C in $\mathcal{L}$ marking the interface between Phase I and II materials. At C,

$$T = T_c, \tag{1b}$$

$$K_1|\text{grad } T^-| = K_2|\text{grad } T^+|, \tag{1c}$$

where T^-, T^+ denote the limiting temperatures at C from within the Phase I and Phase II regions of $\mathcal{L}$, respectively.

We wish to determine a function $T(x, y)$ and curves C, which satisfy (1a, b, c) for given T^0. This can easily be done by a simple nonlinear change of variables. For let

$$K(\beta) = \begin{cases} K_1, & \text{for } \beta \le T_c, \\ K_2, & \text{for } \beta > T_c, \end{cases} \tag{2a}$$

and

$$U(x, y) = \int_0^{T(x,y)} K(\beta) d\beta ; \tag{2b}$$

then

$$U_{xx} + U_{yy} = 0 \tag{2c}$$

in the Phase I and II regions, while at C $|\text{grad } |U^+| = |\text{grad } U^-|$, with U^-, U^+ the limiting values of U on C from within the Phase I and II regions, respectively. Thus U can be considered harmonic throughout $\mathcal{L}$. On Γ,

$$U(\sigma) = U^0(\sigma) = \int_0^{T^0(\sigma)} K(\beta) d\beta . \tag{2d}$$

A harmonic function $U(x, y)$ satisfying (2c, d) exists and can be found using well-known methods of potential theory (see [1]). Since the function K of (2a) never vanishes, one can solve (2b) for the function $T(x, y)$ in terms of $U(x, y)$, which obeys (1a, b, c); moreover the interface curve C on which $T = T_c$ is simply the equipotential curve for U on which

$$U = \int_0^{T_c} K(\beta) d\beta .$$

2. A Related Problem for "Energy." Define T and a function κ as functions of a new variable e by

$$T(e) = \begin{cases} T_c + (e - H)/c_1, & \text{for } e < H ; \\ T_c, & \text{for } H \le e \le 2H ; \\ T_c + (e - 2H)/c_2, & \text{for } e > 2H ; \end{cases} \tag{3}$$

$$\kappa(e) = \begin{cases} \kappa_1, & \text{for } e \le H ; \\ \phi_1(e), & \text{for } H \le e \le H + \epsilon ; \\ \delta, & \text{for } H + \epsilon \le e \le 2H - \epsilon ; \\ \phi_2(e), & \text{for } 2H - \epsilon \le e \le 2H ; \\ \kappa_2, & \text{for } 2H \le e, \end{cases} \tag{4}$$

where ϵ, δ are any (small) positive numbers, and ϕ_1, ϕ_2 are any smooth monotonic functions such that $\kappa(e)$, $\kappa'(e)$ are continuous. Let $E^0(\sigma)$ be defined on Γ in such a way that $T(E^0(\sigma)) = T^0(\sigma)$ (by (3)). Consider the boundary value problem

(5a) $$(\kappa(e)e_x)_x + (\kappa(e)e_y)_y = 0 \quad \text{on} \quad \mathfrak{L} ;$$

(5b) $$e = E^0 \quad \text{on} \quad \Gamma .$$

We claim that an analytic solution of (5a, b) exists, which as ϵ, δ tend to zero, converges to a function yielding by (3) a piecewise harmonic function T obeying (1a, b, c).

Let

(6) $$F(e) = \int_0^e \kappa(\beta)d\beta ;$$

then as a function of x, y, F obeys (by (5a, b))

(6a) $$F_{xx} + F_{yy} = 0 \quad \text{on} \quad \mathfrak{L} ,$$

(6b) $$F(\sigma) = \int_0^{E^0(\sigma)} \kappa(\beta)d\beta \quad \text{on} \quad \Gamma .$$

Under the assumptions on E^0, κ, such a function F exists, and since $F'(e) = \kappa(e) \neq 0$, e and T may be found by (6), (3).

The equipotential curves for F on which e, F are constant, are Jordan arcs joining points of Γ. Let $\mathfrak{L}^-$, $\mathfrak{L}^0$, $\mathfrak{L}^+$ be the subsets of $\mathfrak{L}$ in which $e < H$, $H < e < 2H$, $e > 2H$, respectively. These regions are bounded by smooth Jordan arcs or sets of curves C^H, C^{2H} in $\mathfrak{L}$ on which $e = H$, $F = \kappa_1 H$, and $e = 2H$, $F = \kappa_1 H + \int_H^{2H} \kappa(\beta)d\beta$, respectively. The regions clearly depend on ϵ, δ.

Let ϵ tend to zero, with $\kappa(e)$ converging in a decreasing manner to the piecewise constant function

(7) $$\kappa(e) = \begin{cases} \kappa_1 , & \text{for} \quad e \leq H ; \\ \delta , & \text{for} \quad H < e < 2H ; \\ \kappa_2 , & \text{for} \quad e \geq 2H ; \end{cases}$$

by Dini's theorem F converges uniformly on Γ to a continuous function given by (6b) (see [2, p. 106]). Consequently, as ϵ tends to 0, F converges uniformly on $\mathfrak{L} + \Gamma$ to a harmonic function satisfying (6a, b) with κ defined by (7). By Harnack's theorem the domains $\mathfrak{L}^+$, $\mathfrak{L}^0$, $\mathfrak{L}^-$ and curves C^H, C^{2H} converge to domains $\mathfrak{L}^+$, $\mathfrak{L}^0$, $\mathfrak{L}^-$ and smooth Jordan arcs C^H, C^{2H} as above. From (6),

(6c) $$\begin{gathered} F = \kappa_1 e \leq \kappa_1 H \quad \text{on} \quad \mathfrak{L}^- ; \\ \kappa_1 H \leq F = \kappa_1 H + \delta\,(e - H) \leq H(\kappa_1 + \delta) \quad \text{on} \quad \mathfrak{L}^0 ; \\ H(\kappa_1 + \delta) \leq F = H(\kappa_1 + \delta) + \kappa_2(e - 2H) \quad \text{on} \quad \mathfrak{L}^+ , \end{gathered}$$

and

(6d) $$F = \begin{cases} \kappa_1 H \quad \text{on} \quad C^H ; \\ H(\kappa_1 + \delta) \quad \text{on} \quad C^{2H} . \end{cases}$$

From (6c),

$$\text{(8)} \qquad |\text{grad } F| = \begin{cases} \kappa_1 |\text{grad } e| , & \text{in} \quad \mathfrak{L}^- ; \\ (\kappa_1 + \delta) |\text{grad } e| , & \text{in} \quad \mathfrak{L}^0 , \\ \kappa_2 |\text{grad } e| , & \text{in} \quad \mathfrak{L}^+ . \end{cases}$$

Since $F(x, y)$ is constant on C^H it increases as the point (x, y) crosses C^H from $\mathfrak{L}^-$ to $\mathfrak{L}^0$. Let e^-, e^0 be the limiting values of e on C^H from within $\mathfrak{L}^-$ and $\mathfrak{L}^0$ respectively. Since F is harmonic throughout $\mathfrak{L}$,

$$\text{(9a)} \qquad |\text{grad } e^0| (\kappa_1 + \delta) = \kappa_1 |\text{grad } e^-|$$

on C^H (Eq. 8). Similarly, letting e^+, e^0 be the limiting values of e on C^H, C^{2H}, from $\mathfrak{L}^+$, $\mathfrak{L}^0$ respectively,

$$\text{(9b)} \qquad |\text{grad } e^+| \kappa_2 = (\kappa_1 + \delta) |\text{grad } e^0| ;$$

thus by (6), e is harmonic within $\mathfrak{L}^0$, $\mathfrak{L}^-$, $\mathfrak{L}^+$, continuous throughout $\mathfrak{L}$, and has a discontinuous gradient across C^H, C^{2H}.

Let δ tend to 0. By reasoning similar to that above, $\kappa(e)$ converges to the step function

$$\text{(10)} \qquad \kappa(e) = \begin{cases} \kappa_1 , & \text{for} \quad e \leq H ; \\ 0 , & \text{for} \quad H < e < 2H ; \\ \kappa_2 , & \text{for} \quad e \geq 2H , \end{cases}$$

and F converges uniformly on $\mathfrak{L}^+$ and Γ to a harmonic function obeying (6a, b). Moreover by (5d), C^H, C^{2H} converge to a common curve C, on which $F = H\kappa_1$, while $\mathfrak{L}^0$ tends to the empty set, and $\mathfrak{L}^+$, $\mathfrak{L}^-$ tend to sets $\mathfrak{L}^{\mathrm{II}}$, $\mathfrak{L}^{\mathrm{I}}$, on which $F > \kappa_1 H$ and $F < \kappa_1 H$ respectively. The functions e converge to a function harmonic on $\mathfrak{L}^{\mathrm{II}}$ and $\mathfrak{L}^{\mathrm{I}}$, and

$$\text{(11)} \qquad F = \begin{cases} \kappa_1 e , & \text{on} \quad \mathfrak{L}^{\mathrm{I}} ; \\ \kappa_1 H + \kappa_2 (e - 2H) , & \text{on} \quad \mathfrak{L}^{\mathrm{II}} . \end{cases}$$

For e^+, e^- the limiting values of e at C from within $\mathfrak{L}^{\mathrm{II}}$, $\mathfrak{L}^{\mathrm{I}}$, respectively,

$$\text{(12)} \qquad e^- = H, e^+ = 2H , \quad \text{on} \quad C .$$

Moreover, since grad F is continuous over $\mathfrak{L}$,

$$\kappa_1 |\text{grad } e^-| = \kappa_2 |\text{grad } e^+| , \quad \text{on} \quad C$$

and $e = E^0$ on Γ. Using (3), we now obtain a function $T(x, y)$ which is harmonic on $\mathfrak{L}^{\mathrm{I}}$, $\mathfrak{L}^{\mathrm{II}}$ and obeys (1a, b, c). T is the solution to the steady state problem.

Since in the limit for $\epsilon = \delta = 0$, F (and thus T) is determined uniquely by the given boundary values of T on Γ, T is uniquely determined.

3. A Related Variational Problem. As a harmonic function continuous on Γ, F is under suitable conditions the solution to the problem of minimizing the Dirichlet integral over $\mathfrak{L}$ among all functions obeying (6b) (see [1]). This implies by (3), (6), that the solution T to (1a, b, c) is that function minimizing the integral

$$I = \frac{1}{\rho^2} \int \int_{\mathfrak{L}} (K(T))^2 (T_x{}^2 + T_y{}^2) dx dy$$

with $K(T)$ defined by (2a), among all piecewise smooth functions T obeying (1a).

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1. R. COURANT, *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces*, Interscience, New York, 1950. MR **12**, 90.

2. R. COURANT, *Differential and Integral Calculus*, Vol. II, Interscience, New York, 1964.

3. M. ROSE, "A method for calculating solutions of parabolic equations with a free boundary," *Math. Comp.*, v. 14, 1960, pp. 249–256. MR **22** #6085.

4. A. SOLOMON, "Some remarks on the Stefan problem," *Math. Comp.*, v. 20, 1966, pp. 347–360.