A Fundamental Solution for a Biharmonic Finite-Difference Operator[1]

By R. Bruce Simpson

1. Introduction. Several authors have described a fundamental solution for the five-point finite-difference operator which approximates the Laplacian differential operator in the plane, and its asymptotic relation to a fundamental solution of the Laplacian has been known for some time [3], [7]. Recently an explicit bound for the difference between these two functions has been given by Mangad [6]. In a paper, [5], in which he estimates the difference between the Green's functions of the Dirichlet problem over a rectangle for the Laplace differential operator, and a Laplace difference operator, Laasonen shows how such estimates may be used to derive convergence rates for finite-difference approximations to Poisson's equation under very mild restrictions on the inhomogeneous term. It is the object of this paper to establish similar estimates to those of Mangad's for fundamental solutions of the biharmonic differential and difference operators which will enable analyses similar to those of Laasonen's to be made for biharmonic boundary value problems [8]. We consider any bounded region of the plane, and a square grid of mesh size h covering the region. We construct a fundamental solution for the biharmonic operator in the region, and by an analogous procedure, we construct a discrete fundamental solution, defined at the grid points in the region, for the thirteen-point finite-difference operator which approximates the biharmonic operator with truncation error of order 2 [4]. The constructions are made so as to enable us to estimate the difference between these two functions as the mesh spacing varies.

By first extending slightly the estimate of Mangad to give a bound for the difference between the first divided differences of the continuous and discrete fundamental solutions to the corresponding Laplacian operators, we can obtain a similar estimate for the convergence of the first differences of the discrete biharmonic fundamental solution to the differences of the continuous one. The manner of extending these results to certain polyharmonic difference operators will be apparent from the constructions used here.

2. Preliminaries. Points of the plane, the set E_2, will be denoted by vectors x, with coordinates in a rectangular coordinate system (x_1, x_2), and the length of x will be given by $|x| = (x_1^2 + x_2^2)^{1/2}$. We shall indicate the mesh points of a square grid of mesh size h covering the plane and such that the coordinate axes are grid lines by E_h, and the points of E_h will be denoted by vectors, P, written as capital letters. For a region D of the plane, we define a corresponding set of grid points by $D_h \equiv \bar{D} \cap E_h$. $C_\epsilon(x)$ is to be the open disc of radius ϵ centered on x, and $S_a(x)$ is the square of side length a, oriented as parallel to the grid squares, and centered

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on x. For a function $V(P_1, P_2)$, defined on E_h, we introduce the first divided differences

$$\delta_{hP_1} V(P_1, P_2) \equiv \frac{V(P_1 + h, P_2) - V(P_1, P_2)}{h}$$

and

$$\bar{\delta}_{hP_2} V(P_1, P_2) \equiv \frac{V(P_1, P_2) - V(P_1, P_2 - h)}{h}$$

and the discrete Laplacian difference operator

$$\Delta_h V(P) = \sum_{i=1}^{2} \bar{\delta}_{hP_i} \delta_{hP_i} V(P) . \tag{1}$$

The set $N_1(P) \equiv \{Q | Q \in E_h, |P - Q| \leq h\}$ is the set of arguments at which $V(Q)$ is required to form $\Delta_h V(P)$. For a set of grid points D_h, $N_1(D_h) \equiv \{Q | Q \in E_h, Q \in N_1(P)$ for some $P \in D_h\}$. The thirteen-point discrete biharmonic difference operator [4], denoted by Δ_h^2, can be defined by adopting a property of its continuous counterpart, Δ^2, i.e.

$$\Delta_h^2 V(P) \equiv \Delta_h(\Delta_h V(P)); \tag{2}$$

evidently $N_2(P) \equiv N_1(N_1(P))$ is the set of arguments at which $V(Q)$ is required to form $\Delta_h^2 V(P)$.

While the following device for estimating certain discrete sums has been used by Bramble and Hubbard, [2], there appears to be no explicit reference for it.

LEMMA 1. *Let $f(x)$ be a nonnegative function, integrable over a region R' in E_2, and subharmonic in a subregion R. Then*

$$h^2 \sum_{T \in D_h} f(T) \leq \frac{4}{\pi} \int_{R'} f dA$$

for any region D such that $T \in D_h$, $C_{h/2}(T) \subseteq R \subseteq R'$.

Proof. The proof is an application of the solid mean-value inequality for subharmonic functions, which in two dimensions is

$$f(P) \leq \frac{1}{\pi r^2} \int_{C_r(P)} f dA .$$

In particular, taking $r = h/2$ and $P \in D_h$

$$f(P) \leq \frac{4}{\pi h^2} \int_{C_{h/2}(P)} f dA \leq \frac{4}{\pi h^2} \int_{S_h(P)} f dA ; \tag{3}$$

multiplying (3) by h^2, and summing over D_h proves the result. We will use multi-indices $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = \alpha_1 + \alpha_2$ and the symbol $D^\alpha f = \partial^{|\alpha|} f(x) / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$ to describe derivatives.

LEMMA 2. *Let $f(x)$ be harmonic in a region R, so that $f(x) = \operatorname{Re} F(z)$, where $F(z)$ is an analytic function of $z = x_1 + ix_2$ in R. Then*

$$|D^\alpha f| \leq \left| \frac{d^{|\alpha|} F(z)}{dz^{|\alpha|}} \right| \quad in \quad R .$$

Proof. First let us consider the case when $\alpha = (\alpha_1, 0)$. Then, since the partial derivatives are taken along the real axis,

$$\frac{\partial^{\alpha_1} \operatorname{Re} F(z)}{\partial x_1^{\alpha_1}} = \operatorname{Re} \frac{d^{\alpha_1} F(z)}{dz^{\alpha_1}}$$

so that

$$|D^\alpha f| = \left| \operatorname{Re} \frac{d^{\alpha_1} F(z)}{dz^{\alpha_1}} \right| \leqq \left| \frac{d^{\alpha_1} F(z)}{dz^{\alpha_1}} \right| .$$

Now let $g(x) = \operatorname{Im} F(z) = \operatorname{Re}(-iF(z))$ in R. Then, for α_2 odd, $\alpha_2 = 2m - 1$,

$$\frac{\partial^{\alpha_2} f}{\partial x_2^{\alpha_2}} = (-1)^m \frac{\partial^{\alpha_2} g}{\partial x_1^{\alpha_2}}$$

and

$$\left| \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right| = \left| \frac{\partial^{|\alpha|} g}{\partial x_1^{|\alpha|}} \right| \leqq \left| \frac{d^{|\alpha|} - iF(z)}{dz^{|\alpha|}} \right| \leqq \left| \frac{d^{|\alpha|} F(z)}{dz^{|\alpha|}} \right| .$$

On the other hand, for α_2 even, $\alpha_2 = 2m$,

$$\frac{\partial^{\alpha_2} f}{\partial x_2^{\alpha_2}} = (-1)^m \frac{\partial^{\alpha_2} f}{\partial x_1^{\alpha_2}}$$

and, again,

$$\left| \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right| = \left| \frac{\partial^{|\alpha|} f}{\partial x_1^{|\alpha|}} \right| \leqq \left| \frac{d^{|\alpha|} F(z)}{dz^{|\alpha|}} \right| .$$

From this lemma, we conclude directly that for $x \neq s$, there is a constant $k(\alpha)$, depending only on $|\alpha|$, such that

$$|D_x^\alpha \log |x - s|| \leqq \frac{k(\alpha)}{|x - s|^{|\alpha|}} ;$$

In deriving the estimate of this paper, we shall use k and K to denote generic constants which are independent of h (their appearance in successive inequalities does not represent the same value, but only the fact that there is some constant for which the inequality is valid).

Let $A(P; Q) = A(P_1, P_2; Q_1, Q_2)$ be a function of two grid points P and Q, and let L_h stand for either Δ_h or Δ_h^2. Then $A(P; Q)$ is defined to be a fundamental solution for L_h on the set of grid points $D_h \subset E_h$ if it is defined on $N^2(D_h) \times N_2(D_h)$, and satisfies both

$$\left.\begin{aligned} L_{hP} A(P; Q) &= \frac{\delta(P; Q)}{h^2} ; \\ L_{hQ} A(P; Q) &= \frac{\delta(P; Q)}{h^2} , \end{aligned}\right. \qquad P \in D_h ;\ Q \in D_h .$$

Here $\delta(P; Q)$ denotes the Kronecker delta symbol, and the subscript on the operator indicates that the operator acts on $A(P; Q)$ as a function of that variable which depends parametrically on the other.

It is shown by McCrea and Whipple [7], that the function defined for m and n integers

$$g(m, n) \equiv \frac{1}{2\pi} \int_0^\pi \frac{1 - \exp(-|m|y) \cos ns}{\sinh y} ds \tag{4}$$

where y lies on the branch of the root of $\cos s + \cosh y = 2$ which varies between 0 and $|\cosh^{-1} 3|$ could be used to define a fundamental solution, $\Gamma_h(P; Q)$, for $-\Delta_h$ in E_h;

$$\Gamma_h(P; Q) \equiv g\left(\frac{P_1 - Q_1}{h}, \frac{P_2 - Q_2}{h}\right) - \frac{3}{4\pi} \log 2 - C_e \tag{5}$$

where C_e is Euler's constant. If we denote the radially symmetric fundamental solution of the Laplacian $(-\Delta)$ by

$$\gamma(x; \xi) = \frac{1}{2\pi} \log |x - \xi| \tag{6}$$

then Mangad has shown [6] that

$$|\gamma(P; Q) - \Gamma_h(P; Q)| \leqq 54(h/|P - Q|)^2 . \tag{7}$$

Consider a bounded subset D of E_2, and a square Σ the sides of which are grid lines and such that $\bar{D}$ lies in the interior of Σ; and let $G_h(P; Q)$ be defined on $\Sigma_h \times \Sigma_h$, for any $Q \in \Sigma_h$, by

$$-\Delta_{hP} G_h(P; Q) = \frac{\delta(P; Q)}{h^2}, \qquad P \in \Sigma_h - (\partial\Sigma)_h ,$$

$$G_h(P; Q) = 0 , \qquad P \in (\partial\Sigma)_h$$

(i.e., $G_h(P; Q)$ is the discrete Green's function for Σ_h). From (7), it can be seen that we can choose positive constants k_0 and k_1, depending on the distance d from $\partial\Sigma$ to $\bar{D}$, such that for $Q \in D_h$, $h < d/3$, $P \in (\partial\Sigma)_h$

$$H_0(P; Q) \equiv -G_h(P; Q) - k_0 + \Gamma_h(P; Q) \leqq 0 ,$$

$$H_1(P; Q) \equiv -G_h(P; Q) + k_1 + \Gamma_h(P; Q) \geqq 0 .$$

Since $\Delta_{hP} H_i(P; Q) = 0$, $i = 0, 1$, for $P \in \Sigma_h - (\partial\Sigma)_h$, by the discrete maximum-minimum principle [1],

$$G_h(P; Q) - k_1 \leqq \Gamma_h(P; Q) \leqq G_h(P; Q) + k_0$$

i.e.,

$$|\Gamma_h(P; Q)| \leqq G_h(P; Q) + k_2 , \qquad k_2 = \max(k_0, k_1) .$$

Laasonen [5] has shown that for $a \equiv$ side length of Σ

$$G_h(P; Q) \leqq G_h(Q; Q) \leqq .354 (\log(a/h + 1))$$

which allows us to state

LEMMA 3. *For a given h_0 and bounded subset D of E_2, there is a constant k_2 depend-*

ing on h_0 and D only, such that for $h < h_0$,

$$\max_{P,Q\in D_h} |\Gamma_h(P;Q)| \leqq .354|\log h| + k_2 .$$

The technique of Mangad, [6], in obtaining (7), can easily be modified to prove the following

THEOREM 1. *There exists a numerical constant c such that for $|P - Q| > h$,*

$$|\delta_h\Gamma_h(P;Q) - \delta_h\gamma(P;Q)| \leqq ch^2/|P - Q|^3$$

where δ_h denotes any first difference with respect to either P_i or Q_i, $i = 1, 2$.

We shall indicate the necessary modifications of Mangad's proof of (7), and we observe that while our outline is simplified to show only the existence of c, a more lengthy calculation could be performed in a similar manner to provide c explicitly. Since

$$\begin{aligned} \delta_{hQ_i}\Gamma_h(P;Q) &= -\bar{\delta}_{hP_i}\Gamma_h(P;Q) , \\ \delta_{hQ_i}\gamma(P;Q) &= -\bar{\delta}_{hP_i}\gamma(P;Q) , \end{aligned} \tag{9}$$

we may consider only differences with respect to P. Furthermore, from (5), and the symmetries of $g(m, n)$ [7] it can be seen that it is sufficient to consider $Q = (0, 0) \equiv 0$, and P to be in the sector of the plane characterized by

$$0 \leqq P_2 \leqq P_1 , \qquad P \not\equiv (0, 0) \quad \text{or} \quad (h, 0) . \tag{10}$$

However, it appears to be necessary to consider the P_1 and P_2 differences separately. From a standard Laplace transform, for $P_i = m_i h$,

$$\delta_{hP_2}\gamma(P;0) = \frac{1}{2\pi h}\int_0^\infty \left(\frac{\cos m_2 s - \cos(m_2+1)s}{s}\right)e^{-m_1 s}ds .$$

Hence, with (4), we have, for $\cos s + \cosh y = 2$

$$\begin{aligned} 2\pi|\delta_{hP_2}(\Gamma_h - \gamma)(P;0)| = \frac{1}{h}\Bigg|\int_0^\pi &\left(\frac{\cos m_2 s - \cos(m_2+1)s}{\sinh y}\right)e^{-m_1 y}ds \\ &- \int_0^\infty (\cos m_2 s - \cos(m_2+1)s)\frac{e^{-m_1 s}}{s}ds\Bigg| . \end{aligned} \tag{11}$$

(Here, and in the proofs to follow, we use the common notation $(f \pm g)(x)$ for the function $f(x) \pm g(x)$.)

We break up the right side of (11) as

$$2\pi|\delta_{hP_2}(\Gamma_h - \gamma)(P;0)| \leqq |A| + |B| + |C| , \tag{12}$$

where

$$A = \frac{1}{h}\int_\epsilon^\infty \left(\frac{\cos m_2 s - \cos(m_2+1)s}{s}\right)e^{-m_1 s}ds ,$$

$$B = \frac{1}{h}\int_\epsilon^\pi \left(\frac{\cos m_2 s - \cos(m_2+1)s}{\sinh y}\right)e^{-m_1 y}ds ,$$

$$C = \frac{1}{h}\int_0^\epsilon (\cos m_2 s - \cos(m_2+1)s)\left(\frac{e^{-m_1 y}}{\sinh y} - \frac{e^{-m_1 s}}{s}\right)ds ,$$

for some choice of ϵ in $0 < \epsilon < 1$, independent of h. The estimation of terms $|A|$ and $|B|$ of (12) is entirely analogous to that in [6]; we indicate how $|C|$, the more complicated term, is estimated. From Lemma 2 of [6], for $0 \leqq s \leqq 1$, $s - s^3/10 \leqq y \leqq s$, and a straightforward calculation shows that $\lim_{s\to 0+} (\sinh y)/s = (\frac{2}{3})^{1/2}$. Hence for some $\epsilon > 0$

$$\frac{e^{-m_1 y}}{\sinh y} \geqq \frac{e^{-m_1 s}}{s} \quad \text{when} \quad 0 < s \leqq \epsilon \tag{13}$$

and we have

$$\begin{aligned} |C| &\leqq \frac{1}{h}\int_0^\epsilon s\left|\frac{e^{-m_1 y}}{\sinh y} - \frac{e^{-m_1 s}}{s}\right| ds \\ &\leqq \frac{1}{h}\int_0^\epsilon \left(\frac{\exp m_1 \dfrac{s^3}{10}}{1 - \dfrac{s^2}{10}} - 1\right) e^{-m_1 s} ds \qquad (14) \\ &\leqq \frac{10}{(10-\epsilon^2)h}\int_0^\epsilon \left(\exp\left(\frac{m_1 s^3}{10}\right) - 1 + \frac{s^2}{10}\right) e^{-m_1 s} ds \\ &\leqq \frac{1}{(10-\epsilon^2)h}\left(\int_0^\epsilon m_1 s^3 \exp\left(-m_1\left(s - \frac{s^3}{10}\right)\right) ds + \int_0^\epsilon s^2 e^{-m_1 s} ds\right). \end{aligned}$$

Now on $(0, \epsilon)$ $s - s^3/10 \geqq (1 - \epsilon^2/10)s$, so that

$$\exp(-m_1(s - s^3/10)) \leqq \exp(-(1 - \epsilon^2/10)m_1 s)$$

and

$$\begin{aligned} |C| &\leqq \frac{m_1}{(10-\epsilon^2)h}\int_0^\infty s^3 \exp\left(-\left(1 - \frac{\epsilon^2}{10}\right)m_1 s\right) ds + \frac{h^2}{(10-\epsilon^2)m_1^3 h^3} \\ &\leqq \frac{m_1}{(10-\epsilon^2)h}\left(\frac{1}{(1-\epsilon^2/10)m_1}\right)^4 + k(\epsilon)\frac{h^2}{|P|^3} \leqq k(\epsilon)\frac{h^2}{|P|^3} \end{aligned}$$

since from (10), $P_1 = m_1 h \geqq |P|/2^{1/2}$. We note that the more complicated procedure of [6] would avoid the uncertainty about the range of ϵ introduced at (13) and would permit explicit estimation of c.

Similarly, we have

$$2\pi|\delta_{hP_1}(\Gamma_h - \gamma)(P; 0)| \leqq |A'| + |B'| + |C'|$$

for

$$\begin{aligned} A' &= \frac{1}{h}\int_\epsilon^\infty \frac{\cos m_2 s}{s}(e^{-m_1 s} - e^{-(m_1+1)s}) ds\,, \\ B' &= \frac{1}{h}\int_\epsilon^\pi \frac{(1-e^{-y})e^{-m_1 y}}{\sinh y}\cos m_2 s ds\,, \\ C' &= \frac{1}{h}\int_0^\epsilon \cos m_2 s\left(\frac{1-e^{-y}}{\sinh y}e^{-m_1 y} - \frac{1-e^{-s}}{s}e^{-m_1 s}\right) ds \end{aligned}$$

and, again, we will only indicate the treatment of C'. If I is used to denote the integrand of C', we let S^+ (S^-) be the subset of $(0, \epsilon)$ on which I is positive (negative). Since $\sinh y \geqq y \geqq s - s^3/10$, $s \geqq y$ for $\epsilon < 1$ [6, Lemma 2], on S^+

$$|I| \leqq \left(\frac{1 - e^{-s}}{s - s^3/10}\right)e^{-m_1 y} - \left(\frac{1 - e^{-s}}{s}\right)e^{-m_1 s}$$

$$\leqq \frac{1 - e^{-s}}{s}\left(\frac{\exp\left(-m_1\left(s - \frac{s^3}{10}\right)\right)}{1 - s^2/10} - e^{-m_1 s}\right)$$

$$\leqq \frac{k}{1 - \epsilon^2/10} e^{-m_1 s}\left(\exp\left(\frac{m_1 s^3}{10}\right) - 1 - \frac{s^2}{10}\right).$$

Hence

$$\frac{1}{h}\int_{S^+} |I| ds \leqq \frac{k}{h}\int_0^\epsilon \left(\exp\left(\frac{m_1 s^3}{10}\right) - 1 + \frac{s^2}{10}\right)e^{-m_1 s} ds$$

which was estimated in (14). On S^-

$$|I| \leqq \frac{1 - e^{-s}}{s} e^{-m_1 s} - \frac{1 - e^{-y}}{\sinh y} e^{-m_1 s};$$

now $\sinh y \leqq \sinh s \leqq s + (s^3/3!) \cosh \epsilon$ and $1/\sinh y \geqq (1 - bs^2)/s$ where $b = [\cosh \epsilon]/3!$. Thus on S^-

$$|I| \leqq \left\{\frac{1 - e^{-s}}{s} - \left(1 - \exp\left(-s + \frac{s^3}{10}\right)\right)\left(\frac{1 - bs^2}{s}\right)\right\}e^{-m_1 s}$$

$$\leqq \left(e^{-s}\left(\frac{\exp(s^3/10) - 1}{s}\right) + bs\left(1 - \exp\left(-s + \frac{s^3}{10}\right)\right)\right)e^{-m_1 s}$$

$$\leqq \left(\frac{s^2}{10} + bs^2\right)\exp\left(-s + \frac{s^3}{10}\right)e^{-m_1 s} \leqq ks^2 e^{-m_1 s}$$

for $0 \leqq s \leqq \epsilon < 1$, and

$$\frac{1}{h}\int_{S^-} |I| ds \leqq \frac{1}{h}\int_0^\infty ks^2 e^{-m_1 s} ds. \tag{15}$$

However, the right-hand side of (15) was also estimated at (14). These, then, with the corresponding estimates for $|A|$, $|B|$, $|A'|$, and $|B'|$ and the symmetries of Γ_h, and γ conclude the estimate.

3. A Fundamental Solution for the Biharmonic Differential Operator. To define a function on $E_2 \times E_2$ which is a fundamental solution for Δ^2 in a bounded region D, let L be a circle centered on the origin and containing D, and let L_0 be a circle with the same center but with radius r_0 equal to twice the radius of L. L_1 is to be a larger circle than L_0, centered on the origin, with radius r_1. Let $f(s) \in C_0^\infty(I)$, where I denotes the real line, $0 \leqq f(s) \leqq 1$, and $f(s) \equiv 1$ for $0 \leqq s \leqq r_0$, $f(s) \equiv 0$ for $s \geqq r_1$; we define $\eta(x)$ on E_2 by $\eta(x) = f(|x|)$ and we define $B(x; \xi)$ on $E_2 \times E_2$ as

$$B(x; \xi) \equiv \int \eta(t)\gamma(x; t)\gamma(\xi; t) dA_t.$$

For convenience in the sequel, we will assume that $r_1 = 2r_0$, and we summarize some of the properties of this function in the following theorem.

THEOREM 2. (i) *$B(x;\xi)$ is a fundamental solution for Δ^2 in D.*

(ii) *$B(x;\xi) \in C^1(E_2 \times E_2)$ and a modulus of continuity for any of its first derivatives is $\omega(\delta) = K\delta\,|\log\delta|$.*

(iii) *For $x \neq \xi$, $x, \xi \in L_0$,*

$$\Delta_x B(x;\xi) = \Delta_\xi B(x;\xi) = \gamma(x;\xi)\ .$$

(iv) *For $x \neq \xi$, $B(x;\xi)$ is an infinitely differentiable function of x and ξ, and its derivatives are continuous in the sense of a function of four variables when x and ξ vary in disjoint subsets of L_0.*

(v) *There is a constant K, depending on r_0 and α, such that for $x \neq \xi$ $x, \xi \in L_0$, $|\alpha| \geqq 2$,*

$$|D_x^\alpha B(x;\xi)| \leqq \frac{K(|\log|x-\xi|| + 1)}{|x-\xi|^{|\alpha|-2}}\ .$$

Proof. (i) To see that

$$\int B(x;\xi)\Delta^2\phi(\xi)dA_\xi = \phi(x)$$

for all $\phi(x) \in C_0^\infty(D)$, we need only justify the interchange of order of integration in $\int^\xi\int^t \eta(t)\gamma(x;t)\gamma(\xi;t)\,\Delta^2\phi(\xi)dA_t dA_\xi$ and observe that (a) $\Delta^2\phi(\xi) = -\Delta(-\Delta\phi(\xi))$, (b) $\gamma(\xi;t) = \gamma(t;\xi)$ is a fundamental solution for $-\Delta$, (c) $\eta(t) \equiv 1$ on the support of ϕ.

(ii) Since

$$\left|\int_{C_\epsilon(x)} \eta(t)\frac{1}{|x-t|}\frac{\partial|x-t|}{\partial x_i}\,|\log|\xi-t||dA_t\right|$$

converges to zero as ϵ tends to zero uniformly for $(x,\xi) \in L_0 \times L_0$, we conclude that

$$\frac{\partial B(x;\xi)}{\partial x_i} = \int\eta(t)\frac{\partial\gamma(x;t)}{\partial x_i}\gamma(\xi;t)dA_t \tag{18}$$

exists on $L_0 \times L_0$. Suppose that

$$|(x,\xi) - (x',\xi')| = \left(\sum_{i=1}^2 (x_i - x_i')^2 + (\xi_i - \xi_i')^2\right)^{1/2} < \delta\,,$$

then $x' \in C_\delta(x)$, $\xi' \in C_\delta(\xi)$ and

$$\left|\frac{\partial B(x';\xi')}{\partial x_i} - \frac{\partial B(x;\xi)}{\partial x_i}\right| \leqq |I_1| + |I_2| + |I_3| \tag{19}$$

where, setting $S = E_2 - C_{2\delta}(x) - C_{2\delta}(\xi)$

$$I_1 = \int_S \eta(t)\left(\frac{\partial\gamma(t;x')}{\partial x_i}\gamma(t;\xi') - \frac{\partial\gamma(t;x)}{\partial x_i}\gamma(t;\xi)\right)dA_t$$

and I_2 and I_3 are integrals having the same integrand, but taken over $C_{2\delta}(x)$ and

$C_{2\delta}(\xi)$ respectively. Using Lemma 2, for $(y, w) \in C_\delta(x) \times C_\delta(\xi)$

$$\begin{aligned}
&\left|\int_S \eta(t) \frac{\partial\gamma(y;t)}{\partial t_i} \frac{\partial\gamma(w;t)}{\partial t_j} dA_t\right| \\
(21) \qquad &\leqq \left(\int_S \eta(t)\left(\frac{k}{|y-t|}\right)^2 dA_t\right)^{1/2}\left(\int_S \eta(t)\left(\frac{k}{|w-t|}\right)^2 dA_t\right)^{1/2} \\
&\leqq k\int_\delta^\lambda \frac{dr}{r} \leqq k(\log\lambda - \log\delta)
\end{aligned}$$

for $\lambda = r_0 + r_1$. We can estimate I_1 as follows

$$|I_1| \leqq |A| + |B| ,$$

where, setting $g(t) = \gamma(\xi; t) + \gamma(\xi'; t)/2$, and $sx + (1-s)x' = x(s) \in C_\delta(x)$ for $0 \leqq s \leqq 1$,

$$\begin{aligned}
A &= \int_S \eta(t)g(t)\frac{\partial}{\partial t_i}(\gamma(x;t) - \gamma(x';t))dA_t \\
&= \sum_{k=1}^2 (x_k - x_k')\int_S \eta(t)g(t)\int_0^1 - \frac{\partial^2\gamma(x(s);t)}{\partial t_k \partial t_i} ds dA_t \\
(22) \qquad &= \sum_{k=1}^2 (x_k - x_k')\int_0^1\left(\int_S \frac{\partial\eta(t)g(t)}{\partial t_i}\frac{\partial\gamma(x(s);t)}{\partial t_k}dA_t\right. \\
&\qquad \left. + \oint_{\partial S} n_i\,\eta(t)g(t)\frac{\partial\gamma(x(s);t)}{\partial t_k}d\sigma_t\right)ds ,
\end{aligned}$$

(n_1, n_2) being the outer normal to ∂S. Now, using (21), we can estimate

$$\int_S \frac{\partial\eta(t)g(t)}{\partial t_i}\frac{\partial\gamma(x(s);t)}{\partial t_k}dA_t$$

uniformly for $s \in [0, 1]$, and

$$\begin{aligned}
&\left|\oint_{\partial S} n_i\left(\frac{\gamma(\xi;t) + \gamma(\xi';t)}{2}\right)\frac{\partial\gamma(x(s);t)}{\partial t_k}d\sigma_t\right| \\
&\leqq \frac{k}{\delta}(\log\lambda - \log\delta)\oint_{\partial S} d\sigma_t \leqq K(\log\lambda - \log\delta) .
\end{aligned}$$

Since $|x_k - x_k'| < \delta$, then, $|A| \leqq K\delta(\log\lambda - \log\delta)$. The estimation of B is similar but simpler; i.e. for $\xi(s) = s\xi + (1-s)\xi' \in C_\delta(\xi)$

$$\begin{aligned}
B &= \int_S \frac{\eta(t)}{2}(\gamma(\xi;t) - \gamma(\xi';t))\frac{\partial}{\partial t_i}(\gamma(x;t) + \gamma(x';t))dA_t \\
&= \sum_{k=1}^2\left(\frac{\xi_k - \xi_k'}{2}\right)\int_0^1\int_S \eta(t) - \frac{\partial\gamma(\xi(s);t)}{\partial t_k}\frac{\partial}{\partial t_i}(\gamma(x;t) + \gamma(x';t))dA_t ds .
\end{aligned}$$

The volume integral is estimated using (21) and we have $|B| \leqq K\delta(\log\lambda - \log\delta)$ so that, for $\delta < \delta_0$,

$$|I_1| \leqq K\delta|\log\delta| .$$

To complete our estimate for (10), we must consider I_2 and I_3; however, letting

J be either $C_{2\delta}(x)$ or $C_{2\delta}(\xi)$, it can be verified that, for $x' \in C_\delta(x)$, $\xi' \in C_\delta(\xi)$

$$\left|\int_J \frac{\partial\gamma(x';t)}{\partial x_i}\gamma(\xi';t)dA_t\right| \leqq K\delta|\log\delta| \tag{24}$$

for a uniform constant K, using Hölder's inequality. Using (24) to estimate $|I_2|$ and $|I_3|$, we have, from (19), that for $\delta < \delta_0$, there is a K, dependent on δ_0 and λ only, such that for $|(x, \xi) - (x', \xi')| < \delta$

$$\left|\frac{\partial B(x';\xi')}{\partial x_i} - \frac{\partial B(x;\xi)}{\partial x_i}\right| < K\delta|\log\delta|$$

which proves (ii).

(iii), (iv) Let Λ_1 and Λ_2 be two compact disjoint subsets of E_2, we wish to show that $D_x^\alpha B(x;\xi)$ is continuous on $\Lambda_1 \times \Lambda_2$ for every α. Let $4\epsilon = |\Lambda_1 - \Lambda_2| \equiv \inf_{x\in\Lambda_1;y\in\Lambda_2}|x - y|$ and take Ω to be a piecewise smooth, compact curve enclosing Λ_1 such that $|\Omega - \Lambda_i| \geqq \epsilon$, $i = 1, 2$. Let Z be the interior of Ω, and we assume for convenience that $Z \subset L_0$, i.e. $\eta(t) \equiv 1$, $t \in Z$. From (18), it can be seen that

$$\begin{aligned}\frac{\partial B(x;\xi)}{\partial x_i} &= \int_{E_2-Z}\eta(t)\frac{\partial\gamma(x;t)}{\partial x_i}\gamma(\xi;t)dA_t \\ &\quad + \int_Z \gamma(x;t)\frac{\partial\gamma(\xi;t)}{\partial t_i}dA_t + \oint_\Omega -n_i\gamma(x;t)\gamma(\xi;t)d\sigma_t\end{aligned} \tag{26}$$

where (n_1, n_2) is the outer normal to Ω. For $x \in \Lambda_1$ and $\xi \in \Lambda_2$, if the right side of (26) is differentiated under the integral signs with respect to x_j, the resulting integrals converge uniformly with respect to $x \in \Lambda_1$, i.e., using Gauss's theorem

$$\begin{aligned}\frac{\partial^2 B(x;\xi)}{\partial x_i\partial x_j} &= \int_{E_2-Z}\eta(t)\frac{\partial^2\gamma(x;t)}{\partial x_i\partial x_j}\gamma(\xi;t)dA_t \\ &\quad + \int_Z \gamma(x;t)\frac{\partial^2\gamma(\xi;t)}{\partial t_i\partial t_j}dA_t \\ &\quad + \oint_\Omega n_i\frac{\partial\gamma(x;t)}{\partial t_j}\gamma(t;\xi) - n_j\frac{\partial\gamma(\xi;t)}{\partial t_i}\gamma(x;t)d\sigma_t.\end{aligned} \tag{27}$$

In particular, if we choose $x \neq \xi$, $\Lambda_1 = x$, $\Lambda_2 = \xi$, $\Omega = C_\epsilon(x)$, (27) shows that

$$\begin{aligned}\Delta_x B(x;\xi) &= \int_{E_2-C_\epsilon(x)}\eta(t)\Delta_x\gamma(x;t)\gamma(\xi;t)dA_t \\ &\quad + \int_{C_\epsilon(x)}\gamma(x;t)\Delta_t\gamma(\xi;t)dA_t \\ &\quad + \oint_{|x-t|=\epsilon}\left(\frac{\partial\gamma(x;t)}{\partial n_t}\gamma(\xi;t) - \frac{\partial\gamma(\xi;t)}{\partial n_t}\gamma(x;t)\right)d\sigma_t \\ &= \frac{1}{2\pi\epsilon}\oint_{|x-t|=\epsilon}\gamma(\xi;t)d\sigma_t - \frac{\log\epsilon}{2\pi}\oint_{|x-t|=\epsilon}\frac{\partial\gamma(\xi;t)}{\partial n_t}d\sigma_t = \gamma(\xi;x)\end{aligned}$$

since $\gamma(x;t)$ is harmonic for $x \neq t$, enabling us to employ the mean value theorem for harmonic functions for the first line integral above and to conclude that the second vanishes, proving (iii).

Returning to the case of arbitrary but disjoint Λ_1 and Λ_2, we can continue in the manner in which (27) was obtained to see that for $x \in \Lambda_1 \subset Z$, $\xi \in \Lambda_2$

$$
\begin{aligned}
D_x^{\alpha} B(x; \xi) &= \int_{E_2 - Z} \eta(t) \gamma(\xi; t) D_x^{\alpha} \gamma(x; t) dA_t \\
&\quad + (-1)^{|\alpha|} \int_Z \gamma(x; t) D_t^{\alpha} \gamma(\xi; t) dA_t \\
&\quad + \oint_{\Omega;} \sum_{|\rho+\tau| = |\alpha| - 1} n_1 a_{\rho\tau} D_t^{\rho} \gamma(x; t) D_t^{\tau} \gamma(\xi; t) \\
&\quad + n_2 b_{\rho\tau} D_t^{\rho} \gamma(x; t) D_t^{\tau} \gamma(\xi; t) d\sigma_t
\end{aligned} \tag{28}
$$

where $a_{\rho\tau}$, $b_{\rho\tau}$ are numerical constants depending only on the multi-indices which are their subscripts. Since the integrand of the line integral in (28) is uniformly continuous for $(x, \xi, t) \in \Lambda_1 \times \Lambda_2 \times \Omega$, the line integral is continuous on $\Lambda_1 \times \Lambda_2$. That the volume integrals define functions which are continuous on $\Lambda_1 \times \Lambda_2$ can be seen from the approach taken in proving (ii).

(v) To obtain these estimates for the derivatives of $B(x; \xi)$ when $x \neq \xi$, $x, \xi \in L_0$, we set in the proof of (iv) $x = \Lambda_1$, $\xi = \Lambda_2$ and $\Omega = C_\epsilon(x)$, $\epsilon = |x - \xi|/2$ and estimate the various terms on the right side of (28) using Lemma 2. E.g., setting $|\alpha| = a$,

$$
\begin{aligned}
&\left| \int_{E_2 - C_\epsilon(x)} \eta(t) D_x^{\alpha} \gamma(x; t) \gamma(\xi; t) dA_t \right| \\
&\leqq k(\alpha) \left\{ \int_{E_2 - C_\epsilon(x) - C_\epsilon(\xi)} \eta(t) \frac{1}{|x - t|^a} |\log|\xi - t|| dA_t \right. \\
&\qquad \left. + \int_{C_\epsilon(\xi)} \frac{1}{|x - t|^a} |\log|\xi - t|| dA_t \right\} \\
&\leqq k(\alpha) \left\{ \left(\left| \log \frac{|x - \xi|}{4} \right| + \log \lambda \right) \int_{|x-\xi|/4}^{\lambda} \frac{dr}{r^{a-1}} \right. \\
&\qquad \left. + \frac{4}{3|x - \xi|^a} \int_0^{|x-\xi|/4} r |\log r| dr \right\} \\
&\leqq K \frac{|\log|x - \xi|| + 1}{|x - \xi|^{a-2}}
\end{aligned}
$$

where K depends on α and λ. Similarly

$$
\left| \int_{C_\epsilon(x)} \gamma(x; t) D_t^{\alpha} \gamma(\xi; t) dA_t \right| \leqq K \frac{|\log|x - \xi|| + 1}{|x - \xi|^{a-2}}
$$

and the line integrals can easily be shown to satisfy the same estimate, for some K depending only on α and λ.

4. A Fundamental Solution for the Biharmonic Difference Operator. The definition of $B(x; \xi)$ is immediately suggestive of the following construction. We define $\Gamma_h(P; t)$ to be the extension a.e. of $\Gamma_h(P; Q)$ to $E_h \times E_2$ as

$$
\Gamma_h(P; t) = \Gamma_h(P; Q), \qquad t \in S_h(Q),
$$

and let $B_h(P; Q)$ be defined on $E_h \times E_h$ by

$$B_h(P; Q) = \int \eta(t)\Gamma_h(P; t)\Gamma_h(Q; t)dA_t .$$

THEOREM 3. *$B_h(P; Q)$ is a fundamental solution of Δ_h^2 in L_{0h}.*

Proof. This result follows immediately from applying Δ_{hP}^2 to $B_h(P; Q)$, observing (2) and the fact that $\Gamma_h(P; S)$ is a fundamental solution for $-\Delta_h$.

The apparent fact that $B_h(P; Q)$ is an approximation to $B(P; Q)$ is given quantitative substance by the main results of this paper, Theorems 4 and 5.

THEOREM 4. *For any constants h_0 and l_0 satisfying $2r_0 > l_0 > 6h_0$, there exists a constant M depending on h_0, l_0 and r_0, such that for $h < h_0$,*

$$\text{(i)} \qquad \max_{P,Q \in I_h; |P-Q| > l_0} |B_h(P; Q) - B(P; Q)| \leqq Mh^2(|\log h|)$$

and a constant M_1 depending on h_0 and r_0 such that for $h < h_0$

$$\text{(ii)} \qquad \max_{P,Q \in L_{0h}} |B_h(P; Q) - B(P; Q)| \leqq M_1 h^2(\log h)^2$$

where L_0 is the circle of radius r_0 centered on the origin.

Proof. By definition, we have

$$\begin{aligned} B_h(P; Q) - B(P; Q) &= \int \eta(t)(\Gamma_h(P; t)\Gamma_h(Q; t) - \gamma(P; t)\gamma(Q; t))dA_t \\ &= \int \eta(t)(\Gamma_h - \gamma)(P; t)\left(\frac{\Gamma_h + \gamma}{2}\right)(Q; t)dA_t \\ &\quad + \int \eta(t)(\Gamma_h - \gamma)(Q; t)\left(\frac{\Gamma_h + \gamma}{2}\right)(P; t)dA_t \\ &\equiv I_1 + I_2 \end{aligned}$$

where I_1 and I_2 are defined to be the two integrals on the preceding line. Since they are similar in form, it is sufficient to show how $|I_1|$ can be estimated. We introduce a piecewise constant function

$$\begin{aligned} \gamma_h(P; t) &\equiv \gamma(P; T) , \qquad t \in S_h(T) , T \notin N_2(P) , \\ \gamma_h(P; t) &\equiv \Gamma_h(P; T) , \qquad t \in S_h(T) , T \in N_2(P) . \end{aligned}$$

Then

$$\begin{aligned} I_1 &= \int \eta(t)(\gamma - \gamma_h)(P; t)\left(\frac{\gamma + \Gamma_h}{2}\right)(Q; t)dA_t \\ &\quad + \int \eta(t)(\gamma_h - \Gamma_h)(P; t)\left(\frac{\gamma + \Gamma_h}{2}\right)(Q; t)dA_t \qquad (29) \\ &\equiv J_1 + J_2 . \end{aligned}$$

We wish to use the following observation in estimating J_1. Let $M_{i\phi}$ denote a uniform bound over $S_h(T)$ for the absolute value of the ith derivatives of $\phi(t) \in C^i[S_h(T)]$ and consider $f(t) \in C^2[S_h(T)]$, $g(t) \in C^1[S_h(T)]$. Then, using Taylor's expansions, it is immediate that

$$(30)\qquad \frac{1}{h^2}\left|\int_{S_h(T)} (f(t) - f(T))g(t)dA\right| \leqq \frac{h^2}{3}(M_{2f}|g(T)| + M_{1f}M_{1g}) .$$

If $t \in S_h(T)$, and $T \notin N_2(P)$, then

$$(31)\qquad |P - t| \geqq (2/5)^{1/2}|P - T| ;$$

hence, with Lemma 2,

$$(32)\qquad \begin{aligned} \max_{t\in S_h(T)}\left|\frac{\partial^2\gamma(P;t)}{\partial t_i\partial t_j}\right| &\leqq \frac{k}{|P - T|^2}, \\ \max_{t\in S_h(T)}\left|\frac{\partial\gamma(P;t)}{\partial t_i}\right| &\leqq \frac{k}{|P - T|} . \end{aligned}$$

Noting that the restriction $l_0 > 6h_0$ ensures that $N_2(Q) \cap N_2(P)$ is void, we have with $N_2 \equiv N_2(P) \cup N_2(Q), f(t) = \gamma(P; t), g(t) = \eta(t)((\gamma + \Gamma_h)/2)(Q; t)$

$$(33)\qquad \begin{aligned} |J_1| \leqq h^2 &\sum_{T\in E_h-N_2} \frac{1}{h^2}\left|\int_{S_h(T)} (f(t) - f(T))g(t)dA_t\right| \\ &+ h^2 \sum_{T\in N_2(P)} \frac{1}{h^2}\left(\int_{S_h(T)} |\gamma(P;t)|dA_t + \int_{S_h(T)} |\Gamma_h(P;t)|dA_t\right) \\ &\times\left(\max_{t\in S_h(T)} (|\gamma(Q;t)| + |\Gamma_h(Q;t)|)\right) \\ &+ h^2 \sum_{T\in N_2(Q)} \max_{t\in S_h(T)} \eta(t)|\gamma(P;t) - \gamma(P;T)| \\ &\times\left\{\frac{1}{h^2}\int_{S_h(T)} |\gamma(Q;t)|dA_t + |\Gamma_h(Q;T)|\right\}. \end{aligned}$$

From $|\Gamma_h(Q; T)| \leqq |\gamma(Q; T)| + 54(h^2/|Q - T|^2)$, for $T \notin N_2(Q)$, $|g(T)| \leqq (1/2\pi)|\log|Q - T|| + 6$ and, using (30)–(32), it can be seen that the first term on the right side of (33) is bounded by S_1, where

$$(35)\qquad \begin{aligned} S_1 = kh^2 &\sum_{T\in L_{1h}-N_2} h^2\left(\frac{1}{|P - T|^2}\left(\frac{1}{2\pi}|\log|Q - T|| + 6\right) + \frac{1}{|P - T|}\,\frac{1}{|Q - T|}\right) \\ &+ kh^2 \sum_{r_0\leqq|T|\leqq r_1} h^2\left(\frac{1}{|P - T|}|\log|Q - T||\right)M_{1\eta} . \end{aligned}$$

However, estimating the summand within and without $C_{l_0/2}(Q)$, and using Lemma 1,

$$(36)\qquad \begin{aligned} h^2 \sum_{T\in L_{1h}-N_2} \frac{|\log|Q - T||}{|P - T|^2} &\leqq \left(\log 4r_0 - \log\left(\frac{l_0}{2}\right)\right)h^2 \sum_{T\in L_{1h}-N_2} \frac{1}{|P - T|^2} \\ &\quad + \frac{4}{l_0^2}h^2 \sum_{|T-Q|\leqq l_0/2;T\notin N_2(Q)} |\log|Q - T|| \\ &\leqq k(\log 4r_0 - \log h) + k_1 . \end{aligned}$$

From Schwarz inequality, and Lemma 1

$$(37)\qquad h^2 \sum_{T\in L_{1h}-N_2} \frac{1}{|P - T|}\,\frac{1}{|Q - T|} \leqq k\left(\int_h^{4r_0} \frac{dr}{r}\right) \leqq k(\log 4r_0 - \log h)$$

and it is clear that the remaining terms of (35) can be estimated in a similar fashion, so that for any h_0, there is a constant K depending on h_0, l_0 and r_0 such that

$$S_1 \leqq Kh^2 |\log h| \quad \text{for } h < h_0 .$$

The remaining two terms on the right of (33) are each sums, multiplied by h^2, of 13 terms, where each term is bounded by $K|\log h|$ for $h < h_0$, for a suitable constant K depending on h_0, l_0 and r_0. Hence, for a suitable constant K,

$$|J_1| \leqq Kh^2 |\log h| \quad \text{for} \quad h < h_0 . \tag{38}$$

Using the estimate of $g(T)$ preceding (35), and also (6) and (7), and Lemma 3

$$\begin{aligned} |J_2| \leqq & \sum_{T \in E_h - N_2} \left(54 \frac{h^2}{|P-T|^2}\right) \int_{S_h(T)} \eta(t) \left(\frac{|\log|Q-t||}{2\pi} + 6\right) dA_t \\ & + \sum_{T \in N_2(Q)} 54 \frac{h^2}{|P-T|^2} \int_{S_h(T)} \frac{\eta(t)}{2} \left(\frac{|\log|Q-t||}{2\pi} + k_2 - .354 \log h\right) dA_t \\ \leqq & Kh^2 |\log h| \quad \text{for} \quad h < h_0 \end{aligned} \tag{39}$$

for a suitable constant K. This concludes the estimation of I_1 (Eq. (29)), but, as mentioned, I_2 is similar in form, hence (i) is established.

The second estimate (ii) is obtained by the same process, not using, however, $|P - Q| \geqq l_0$. If we examine the first term of (33), it is bounded by S_1 of (35) which can be estimated uniformly, as in (36) using

$$\begin{aligned} h^2 & \sum_{T \in L_{1h} - N_2} \frac{|\log|Q-T||}{|P-T|^2} \\ & \leqq (\log 4r_0 - \log h) h^2 \sum_{T \in L_{1h} - N_2} \frac{1}{|P-T|^2} \leqq K (\log h)^2 , \end{aligned}$$

and the fact that (37) is already uniform in P and Q. Hence, for P and Q in L_{0h}, and $h < h_0$,

$$S_1 \leqq Kh^2 (\log h)^2$$

for a constant K depending on r_0 and h_0. The remaining two terms in (33) are sums, multiplied by h^2, of 13 terms, each of which is bounded, for $h < h_0$, by $K(\log h)^2$, for a suitable K which depends only on h_0 and r_0. Hence, we have the analogous estimate to (38) uniform in P, Q

$$|J_1| \leqq Kh^2 (\log h)^2 \tag{41}$$

for a suitable constant K. Similar modifications of (39) will show that

$$|J_2| \leqq Kh^2 (\log h)^2$$

for $P, Q \in L_{0h}$ and $h < h_0$, which, with (41), establishes the second estimate.

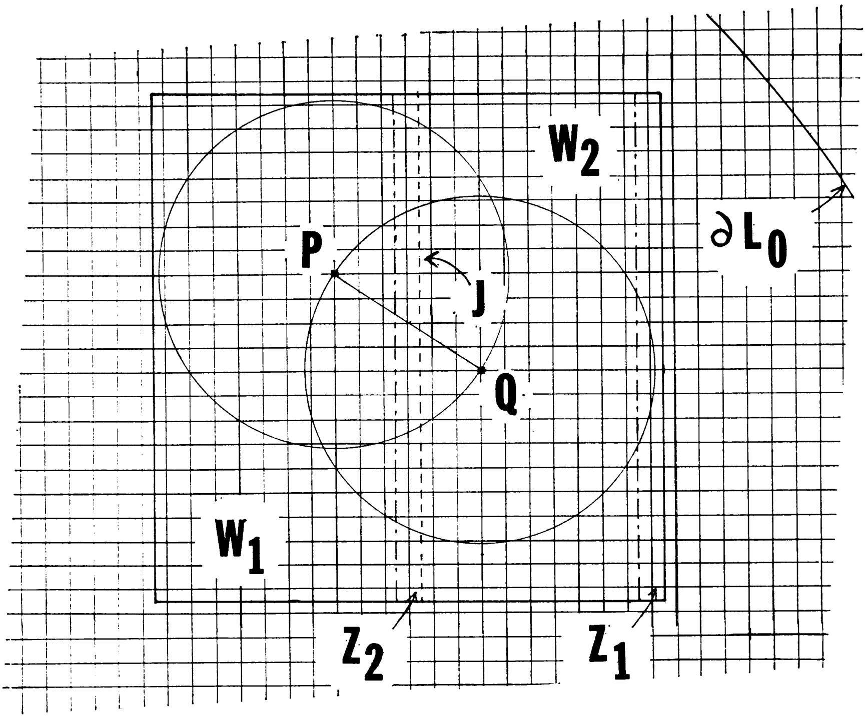
We wish now to conclude our results by using Theorem 1 to establish an estimate similar to (i) of Theorem 4, for the first differences of the fundamental solutions of the biharmonic and discrete biharmonic operators.

THEOREM 5. *For $h_0 < r_0/5$, there exists a constant M_2, depending on h_0 and r_0, such that for any first-difference operator δ_h,*

$$\max_{P,Q \in L_{0h};|P-Q|>5h} |\delta_h(B(P;Q) - B_h(P;Q))| \leqq \frac{M_2 h^2 |\log h|}{|P - Q|}$$

for $h < h_0$.

Proof. While the idea of the proof is essentially the same as that of Theorem 4, some alterations are necessary to provide the nonuniform estimate. For any two points P, $Q \in L_{0h}$ such that $|P - Q| > 5h$, let W be a smallest square containing $S_h(T)$, for any $T \in (C_{|P-Q|}(P) \cup C_{|P-Q|}(Q))_h$, ($C_a(b)$ being the circle of radius a, centered on b as defined above). Consider the grid lines running in a direction which make an angle of 45° or greater with the line segment PQ, and choose a line running in this direction which is halfway between the grid lines of the considered direction and which is one of possibly two such lines that are nearest to the midpoint of PQ. (See Fig. 1.) This line will be labelled J, and coincides with a line of edges of squares $S_h(T)$ which comprise W. It divides W into two rectangles, W_1 the rectangle con-



taining P and W_2 the rectangle containing Q. Using this,

$$(44) \qquad |\delta_h B(P;Q) - \delta_h B_h(P;Q)| \leqq |I_1| + |I_2| + |I_3|$$

where the I_i are integrals with integrand

$$\eta(t)(\delta_h(\Gamma_h(P;t)\Gamma_h(Q;t)) - \delta_h(\gamma(P;t)\gamma(Q;t)))$$

taken over $t \in W_1$, W_2 and $E_2 - W$, respectively. We proceed with the particular case $\delta_h = \delta_{hQ_1}$, the other cases being handled in exactly the same way. With

$$G = \{T | T \in L_{1h},\ T \notin W\},$$

$$|I_3| \leqq \frac{h^2}{2} \sum_{T \in G} \Big\{ |(\Gamma_h + \gamma)(P;T)(\delta_h(\Gamma_h - \gamma)(Q;T))|$$

$$+ |(\Gamma_h - \gamma)(P;T)\delta_h(\Gamma_h + \gamma)(Q;T)|$$

$$+ 2\Big|\gamma(P;T)\delta_h\gamma(Q;T)\frac{1}{h^2}\int_{S_h(T)} \eta(t)dA$$

(45)
$$- 1/h^2 \int_{S_h(T)} \gamma(P;t)\delta_h\gamma(Q;t)\eta(t)dA\Big|\Big\}$$

$$\leqq kh^2\Big(h^2 \sum_{T \in G} \frac{|\log|P - T||}{|Q - T|^3} + \frac{1}{|Q - T|}\frac{1}{|P - T|^2}$$

$$+ \frac{1}{|Q - T|^2}\frac{1}{|P - T|} + M_{1\eta}\Big(\frac{1}{|P - T|}\frac{1}{|Q - T|} + \frac{|\log|P - T||}{|Q - T|^2}\Big)\Big)$$

where we have used Theorem 1, (7) and (30). Using Lemma 1, and the techniques of the preceding proof, we have

(46)
$$|I_3| \leqq kh^2 |\log h|/|P - Q| .$$

Turning to the remaining terms, I_1 and I_2, we can use (9) to see that

$$I_2 = \int_{W_2} \{\Gamma_h(P;t)(\Gamma_h(Q;t_1 - h, t_2) - \Gamma_h(Q;t))/h$$

$$- \gamma(P;t)(\gamma(Q;t_1 - h, t_2) - \gamma(Q;t))/h\}\eta(t)dA_t$$

(47)
$$= \int_{W_2} \{(-\Gamma_h(P;t)\Gamma_h(Q;t) + \gamma(P;t)\gamma(Q;t))/h\}\eta(t)dA_t$$

$$+ \int_{W_3} \{(\Gamma_h(P;t_1 + h, t_2)\Gamma_h(Q;t) - \gamma(P;t_1 + h, t_2)\gamma(Q;t))/h\}$$

$$\times\ \eta(t_1 + h, t_2)dA_t$$

where W_3 is the rectangle obtained by translating W_2 through a distance h in the direction of the negative t_1 axis. We let $W_4 = W_2 \cap W_3$ and let Z_1 and Z_2 be the two rectangles $W_2 - W_4$ and $W_3 - W_4$. We observe that the Z_i have width h and length not exceeding $3|P - Q| + h$, and contain not more than $3|P - Q|/h + 1$ points of E_h. To simplify the following expressions, we shall assume that $W \cup W_3 \subset L_0$, so that $\eta(t) \equiv 1$ on $W \cup W_3$. The modifications of the following estimates which are necessary if $W \cup W_3$ is not contained in L_0 can be seen from the procedures of the preceding proof or the last terms on the right sides of (45). From (47), it is apparent that

$$I_2 = \int_{W_4} \{-\bar{\delta}_{hP_1}\Gamma_h(P;t)\Gamma_h(Q;t) + \bar{\delta}_{hP_1}\gamma(P;t)\gamma(Q;t)\}dA_t$$

(48)
$$+ \frac{1}{h}\int_{Z_1} \{\gamma(P;t)\gamma(Q;t) - \Gamma_h(P;t)\Gamma_h(Q;t)\}dA_t$$

$$+ \frac{1}{h}\int_{Z_2} \{\Gamma_h(P;t_1 + h, t_2)\Gamma_h(Q;t) - \gamma(P;t_1 + h, t_2)\gamma(Q;t)\}dA_t .$$

The integral over W_4 in (48) and I_1 can both be treated in the following manner:

$$
\begin{aligned}
|I_1| \leqq h^2 \sum_{T \in N_2(P)} &\left\{ |\Gamma_h(P; T)\delta_h\Gamma_h(Q; T)| + \frac{1}{h^2}\left| \int_{S_h(T)} \gamma(P; t)\delta_h\gamma(Q; t)dA_t \right| \right\} \\
&+ \frac{h^2}{2} \sum_{T \in W_{1h}-N_2(P)} \left\{ |(\gamma + \Gamma_h)(P; T)\delta_h(\Gamma_h - \gamma)(Q; T)| \right. \\
&+ |(\gamma - \Gamma_h)(P; T)\delta_h(\Gamma_h + \gamma)(Q; T)| + 2\left|\gamma(P; T)\delta_h\gamma(Q; T) \right. \\
&\left.\left. - \frac{1}{h^2}\int_{S_h(T)} \gamma(P; t)\delta_h\gamma(Q; t)dA_t \right| \right\}.
\end{aligned} \tag{49}
$$

For $t \in W_1$, $|Q - t| \geqq |P - Q|/2(2)^{1/2}, - h/2$, i.e.

$$
\begin{aligned}
\frac{1}{|Q - t|} &\leqq \frac{2(2)^{1/2}}{|P - Q|}\left(1 - \frac{(2)^{1/2}h}{|P - Q|}\right)^{-1} \\
&\leqq \frac{10(2)^{1/2}}{|P - Q|}\,\frac{1}{5 - (2)^{1/2}} = \frac{k}{|P - Q|}
\end{aligned} \tag{50}
$$

since $|P - Q| \geq 5h$. To estimate the sum of 13 terms which is the first sum on the right side of (49), we employ Lemma 3, Theorem 1 and (50)

$$
\begin{aligned}
h^2 \sum_{T \in N_2(P)} &\left(|\Gamma_h(P; T)\delta_h\Gamma_h(Q; T)| + \frac{1}{h^2}\left| \int_{S_h(T)} \gamma(P; t)\delta_h\gamma(Q; t)dA_t \right| \right) \\
&\leqq 13\left\{ kh^2(|\log h| + 1)\left(\frac{1}{|P - Q|} + \frac{8ch^2}{|P - Q|^3}\right) \right. \\
&\qquad \left. + \frac{k}{|P - Q|}\int_0^{3h} r(-\log r)dr \right\} \\
&\leqq k\frac{h^2|\log h|}{|P - Q|}
\end{aligned}
$$

for $h < h_0$, k dependent on h_0. For $T \in W_{1h} - N_2(P)$, $|(\gamma + \Gamma_h)(P; T)| \leqq ((1/\pi)\,|\log|P - T|| + 12)$ and

$$
|\delta_h(\Gamma_h + \gamma)(Q; T)| \leqq 2|\delta_h\gamma(Q; T)| + 8c\frac{h^2}{|P - Q|^3} \leqq \frac{k}{|P - Q|};
$$

hence

$$
\begin{aligned}
\frac{h^2}{2} \sum_{T \in W_{1h}-N_2(P)} &\{ |(\gamma + \Gamma_h)(P; T)\delta_h(\Gamma_h - \gamma)(Q; T)| \\
&\qquad + |(\gamma - \Gamma_h)(P; T)\delta_h(\Gamma_h + \gamma)(Q; T)| \} \\
&\leqq \frac{h^2}{2} \sum_{T \in W_{1h}-N_2(P)} \left\{ \left(\frac{1}{\pi}|\log|P - T|| + 12\right)\left(\frac{8ch^2}{|Q - T|^3}\right) \right. \\
&\qquad \left. + \left(\frac{h}{|P - T|}\right)^2 \frac{k}{|P - Q|} \right\} \\
&\leqq k\frac{h^2|\log h|}{|P - Q|}.
\end{aligned} \tag{51}
$$

Using (30) and Lemma 1 in the same manner as previously, it can be seen that the last term of (49) can be bounded by the last term in (51), hence

$$|I_1| \leqq k \frac{h^2}{|P-Q|} |\log h| \tag{52}$$

for a suitable constant k.

As the analysis of I_1 would provide the necessary estimate of the first integral in the expression (48) for I_2, we can complete the estimation of I_2 by considering the integrals over the strips Z_i; e.g. set

$$\begin{aligned} I_4 = \left|\frac{1}{h}\int_{Z_1} \gamma(P;t)\gamma(Q;t) - \Gamma_h(P;t)\Gamma_h(Q;t)dA_t\right| &\leqq \frac{h}{2}\sum_{T\in Z_{1h}} \{|(\gamma-\Gamma_h)(P;T) \\ &\times (\gamma+\Gamma_h)(Q;T)| + |(\gamma-\Gamma_h)(Q;T)(\gamma+\Gamma_h)(P;T)|\} \\ + h\sum_{T\in Z_{1h}} \left|\frac{1}{h^2}\int_{S_h(T)} \gamma(P;t)\gamma(Q;t)dA_t - \gamma(P;T)\gamma(Q;T)\right|&. \end{aligned} \tag{53}$$

Using (7), we can see that the first sum on the right-hand side of (53) is bounded by

$$\begin{aligned} kh^3 \sum_{T\in Z_{1h}} \Bigg\{\frac{1}{|P-T|^2}\left(2|\log|Q-T|| + \frac{54}{25}\right) \\ + \frac{1}{|Q-T|^2}\left(2|\log|P-T|| + \frac{54}{25}\right)\Bigg\}. \end{aligned} \tag{54}$$

For $t \in S_h(T)$, $T \in Z_i$, the triangle inequality gives $|P-t| + h/(2)^{1/2} \geqq |P-T|$, so that for a number a, obtained by the same calculation as gave (50), $1/|P-t| \leqq a/|P-T|$ and similarly $1/|Q-t| \leqq a/|Q-T|$. Thus, from (30), it can be seen that the second sum on the right side of (53) is less than

$$kh^3 \sum_{T\in Z_{1h}} \frac{|\log|P-T||}{|Q-T|^2} + \frac{2}{|P-T||Q-T|} + \frac{|\log|Q-T||}{|P-T|^2}. \tag{55}$$

We observe that

$$\frac{|P-Q|}{2}\left(\frac{1}{(2)^{1/2}} - \frac{1}{5}\right) \leqq \frac{|P-Q|}{2(2)^{1/2}} - \frac{h}{2} \leqq |P-T|$$

and similarly $|Q-T| \geqq k|P-Q|$ when $T \in Z_i$, and using these inequalities and noting the remarks preceding (48) we see that (54) and (55) are bounded for some constant k by $kh^2\,|\log h|/|P-Q|$; e.g.,

$$h^3 \sum_{T\in Z_1} \frac{|\log|P-T||}{|Q-T|^2} \leqq k\frac{h^2|\log|P-Q||}{|P-Q|^2}\, h\sum_{T\in Z_1} 1 \leqq k_1 \frac{h^2|\log h|}{|P-Q|}.$$

Since the integral over Z_2 in (48) can be treated in the same manner, we have, for some k

$$|I_2| \leqq k\frac{h^2|\log h|}{|P-Q|}$$

which with (46), (52) and (44) shows the estimate given in the theorem to be valid when $\delta_h = \delta_{hQ_1}$. As mentioned, however, the other cases are not essentially different and so we shall consider the result proven.

California Institute of Technology
Pasadena, California 91109

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