TECHNICAL NOTES AND SHORT PAPERS

Note on Partitions Modulo 5

By Morris Newman

In the paper preceding this one, Parkin and Shanks study the distribution of the values of the unrestricted partition function $p(n)$ modulo 2 and come to the conclusion that there is no apparent preference, one way or the other, for either the even class or the odd class. Apart from the fact that $p(n)$ is odd infinitely often and even infinitely often (see [3]) nothing of significance is known. In conversation with the author Dr. Shanks asked whether the same is true for partitions modulo 5. In view of the Ramanujan congruence

$$p(5n + 4) \equiv 0 \bmod 5 \tag{1}$$

Some preference for the 0-class is to be expected; but it is not immediately clear that the density (if it exists) of the integers n such that $5|p(n)$ is more than 1/5. We shall show that this is indeed the case. In fact, setting

$$s(x) = \sum_{n \leq x; p(n) \equiv 0 \bmod 5} 1$$

we shall show that

$$\liminf_{x\to\infty} \frac{s(x)}{x} \geq \frac{1}{5} + \frac{36}{5.19^4}. \tag{2}$$

Whether $s(x)/x$ actually approaches a limit as x approaches ∞ seems a difficult question. In fact, the result (2) is deducible only from some deep identities from the theory of the elliptic modular functions. We shall prove (2) by exhibiting arithmetic progressions (disjoint from the arithmetic progression (1)) on which $p(n)$ vanishes modulo 5. In particular we shall show that

$$\begin{aligned} p(5.19^3n + 22006) &\equiv 0 \bmod 5, \qquad n \not\equiv 3 \bmod 19, \\ p(5.19^3n + 15147) &\equiv 0 \bmod 5, \qquad n \not\equiv 7 \bmod 19. \end{aligned} \tag{3}$$

There are of course 36 disjoint arithmetic progressions involved, having a total density of $36/5.19^4$. Each of these progressions is disjoint from the progression in (1), since $22006 \equiv 1 \bmod 5$, $15147 \equiv 2 \bmod 5$.

It will be evident from the proof that follows that (2) is capable of improvement, but we do not pursue this point further. Finally we mention that A. O. L. Atkin has applied similar identities and methods to related problems concerning the partition function and more generally to the coefficients of modular forms, and has obtained some very significant and interesting results.

We turn now to the proof. If n is a nonnegative integer, define $p_r(n)$ as the coefficient of x^n in

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$$\prod_{n=1}^{\infty} (1 - x^n)^r .$$

If n is not a nonnegative integer, define $p_r(n)$ as 0. (Thus $p(n) = p_{-1}(n)$, for example.) A special case of an identity proved by the author in [2] gives the following recurrence formula for $p_{23}(n)$:

$$p_{23}(np^2 + 23\nu) = \gamma_n p_{23}(n) - p^{21} p_{23}((n - 23\nu)/p^2) . \tag{4}$$

Here p is any prime >3, $\nu = (p^2 - 1)/24$, and

$$\gamma_n = c + d\chi_p(23\nu - n)$$

where c, d are independent of n and $\chi_p(n)$ is the Legendre symbol. If n is chosen so that

$$\begin{aligned} n &\equiv 23\nu \bmod p , \\ n &\not\equiv 23\nu \bmod p^2 \end{aligned} \tag{5}$$

then (4) reduces to

$$p_{23}(np^2 + 23\nu) = cp_{23}(n) . \tag{6}$$

The constant c has the value

$$c = p_{23}(23\nu) + p^{10}\chi_p(46\nu) .$$

We now seek a prime $p > 5$ such that $c \equiv 0 \bmod 5$. This entails some numerical work which can be simplified by using formula (8) below, and the smallest permissible choice is $p = 19$. For this choice of p the identity (6) yields the congruence

$$p_{23}(19^2 n + 345) \equiv 0 \bmod 5 , \tag{7}$$

provided that n satisfies the conditions (5).

We now bring in the connection with the partition function. O. Kolberg has shown in [1] that

$$p(n + 1) \equiv p_{23}(n) \bmod 5 , \qquad n \equiv 0, 1 \bmod 5 . \tag{8}$$

If we choose n in (7) so that

$$19^2 n + 345 \equiv 0, 1 \bmod 5$$

then an appropriate application of the Chinese remainder theorem applied to the conditions (5) yields the congruences (3) upon substitution in formula (8).

I would like to thank Dr. Shanks for comments materially improving the presentation of this note.

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1. O. Kolberg, "Some identities involving the partition function," *Math. Scand.*, v. 5, 1957, pp. 77–92. MR **19**, 838.
2. M. Newman, "Further identities and congruences for the coefficients of modular forms," *Canad. J. Math.*, v. 10, 1958, pp. 577–586. MR **20** #4543.
3. M. Newman, "Periodicity modulo m and divisibility properties of the partition function," *Trans. Amer. Math. Soc.*, v. 97, 1960, pp. 225–236. MR **22** #6778.