Generalized Euler and Class Numbers

By Daniel Shanks

1. Introduction. In [1] we discussed the Dirichlet series

(1) $$L_a(s) = \sum_{k=0}^{\infty} \left(\frac{-a}{2k+1}\right) (2k+1)^{-s}$$

where $(-a/(2k+1))$ is the Jacobi symbol. We defined $C_{a,n}$ and $D_{a,n}$ by

(2) $$L_a(2n+1) = \left(\frac{\pi}{a}\right)^{2n+1} \sqrt{a}\, C_{a,n} \qquad L_{-a}(2n) = \left(\frac{\pi}{a}\right)^{2n} \sqrt{a}\, D_{a,n}$$

and showed that these coefficients are *rational* for all $a = 1, 2, 3, \cdots$ and all $n = 0, 1, 2, \cdots$. We also showed how to compute them. We now wish to simplify these coefficients and calculations. Let

(3) $$L_a(2n+1) = \left(\frac{\pi}{2a}\right)^{2n+1} \sqrt{a}\, \frac{c_{a,n}}{(2n)!} \qquad (n = 0, 1, 2, \cdots)$$
$$L_{-a}(2n) = \left(\frac{\pi}{2a}\right)^{2n} \sqrt{a}\, \frac{d_{a,n}}{(2n-1)!} \qquad (n = 1, 2, 3, \cdots) \text{ for } a > 1, \text{ and}$$

(4) $$L_1(2n+1) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+1} \frac{c_{1,n}}{(2n)!} \qquad (n = 0, 1, 2, \cdots)$$
$$L_{-1}(2n) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n} \frac{d_{1,n}}{(2n-1)!} \qquad (n = 1, 2, 3, \cdots).$$

We now assert that the $c_{a,n}$ and $d_{a,n}$ are *integers*. Further, they satisfy simple recurrences on the variable n, and this simplifies their computation.

Consider first a short table of $c_{a,n}$:

	n			
a	0	1	2	3
1	1	1	5	61
2	1	3	57	2763
3	1	8	352	38528
4	1	16	1280	249856
5	2	30	3522	1066590
6	2	46	7970	3487246
7	1	64	15872	9493504
8	2	96	29184	22880256
9	2	126	49410	48649086
10	2	158	79042	96448478

Received March 27, 1967.

The first row are the *Euler* numbers:

(5) $$c_{1,n} = E_n ,$$

which are also called *secant* numbers since

(6) $$\sec w = \sum_{n=0}^{\infty} E_n \frac{w^{2n}}{(2n)!} .$$

The first column are the *class* numbers; that is, there are $c_{a,0}$ inequivalent classes of primitive binary quadratic forms

$$Cu^2 + 2Buv + Av^2$$

with

$$AC - B^2 = a ,$$

the principal form of which is represented by

$$u^2 + av^2 .$$

Our two-dimensional array $c_{a,n}$ therefore generalizes both the Euler numbers and the class numbers—thus our title.

Similarly, a short table of $d_{a,n}$ is shown below. (The number $D_{a,0}$ in (2) actually vanishes for all a, but we do not define $d_{a,0}$.)

	n			
a	1	2	3	4
1	1	2	16	272
2	1	11	361	24611
3	2	46	3362	515086
4	4	128	16384	4456448
5	4	272	55744	23750912
6	6	522	152166	93241002
7	8	904	355688	296327464
8	8	1408	739328	806453248
9	12	2160	1415232	1951153920
10	14	3154	2529614	4300685074

This time the first row consists of the so-called *tangent* numbers

(7) $$d_{1,n} = T_n ,$$

since

(8) $$\tan w = \sum_{n=1}^{\infty} T_n \frac{w^{2n-1}}{(2n-1)!} .$$

2. Recurrences. That these numbers are all integers follows from certain recurrences that they satisfy, and these, in turn, follow from known properties of the *Euler polynomials* $E_n(x)$. We have [2] the generator:

(9) $$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} ,$$

and the known Fourier expansions:

$$(10)\qquad \begin{aligned} E_{2n}(x) &= \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} S_{2n+1}\left(\frac{x}{2}\right), \\ E_{2n-1}(x) &= \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} C_{2n}\left(\frac{x}{2}\right), \end{aligned}$$

where [1, Eq. (18)]

$$(11)\qquad \begin{aligned} S_s(x) &= \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^s}, \\ C_s(x) &= \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^s}. \end{aligned}$$

It follows, if we put

$$x = 2y \quad \text{and} \quad t = 2vi,$$

in (9) that

$$(12)\qquad \begin{aligned} \frac{\pi}{4}\,\frac{\cos v(1-4y)}{\cos v} &= \sum_{n=0}^{\infty} \left(\frac{2v}{\pi}\right)^{2n} S_{2n+1}(y), \\ \frac{\pi}{4}\,\frac{\sin v(1-4y)}{\cos v} &= \sum_{n=1}^{\infty} \left(\frac{2v}{\pi}\right)^{2n-1} C_{2n}(y). \end{aligned}$$

Now, clearly,

$$(13)\qquad L_1(s) = S_s(\tfrac{1}{4}) \quad \text{and} \quad L_{-1}(s) = C_s(0),$$

so that from (12) and (4), together with (6) and (8), we find that $c_{1,n}$ and $d_{1,n}$ are indeed the secant and tangent numbers, respectively.

If a is divisible by a square > 1:

$$(14)\qquad a = bm^2$$

with b square-free, we have [1, Eq. (23)]

$$(15)\qquad L_a(s) = L_b(s) \prod_{p_i | m} \left[1 - \left(\frac{-b}{p_i}\right) p_i^{-s}\right],$$

the product being taken over all odd primes p_i (if any) that divide m.

It follows, from (3), that

$$(16)\qquad \begin{aligned} c_{a,n} &= m^{2n}\left[m \prod_i p_i^{-1}\right]^{2n+1} \prod_i \left[p_i^{2n+1} - \left(\frac{-b}{p_i}\right)\right] c_{b,n}, \\ d_{a,n} &= m^{2n-1}\left[m \prod_i p_i^{-1}\right]^{2n} \prod_i \left[p_i^{2n} - \left(\frac{b}{p_i}\right)\right] d_{b,n}, \end{aligned}$$

if $b > 1$, and, from (4), that

$$(17)\qquad \begin{aligned} c_{m^2,n} &= \frac{1}{2} m^{2n}\left[m \prod_i p_i^{-1}\right]^{2n+1} \prod_i \left[p_i^{2n+1} - \left(\frac{-1}{p_i}\right)\right] c_{1,n}, \\ d_{m^2,n} &= \frac{1}{2} m^{2n-1}\left[m \prod_i p_i^{-1}\right]^{2n} \prod_i \left[p_i^{2n} - 1\right] d_{1,n}, \end{aligned}$$

if $b = 1$. In any case, the $c_{a,n}$ and $d_{a,n}$ are integral multiples of the $c_{b,n}$ and $d_{b,n}$, respectively.

It remains, then, to compute $c_{b,n}$ and $d_{b,n}$ for square-free $b > 1$. We showed, in [1], that for such b we have

$$(18)\qquad \begin{aligned} L_b(2n+1) &= \frac{2}{\sqrt{b}} \sum_k \epsilon_k S_{2n+1}(y_k) , \\ L_{-b}(2n) &= \frac{2}{\sqrt{b}} \sum_k \epsilon_k C_{2n}(y_k) , \end{aligned}$$

where in the linear combinations on the right the ϵ_k are Jacobi symbols, and the y_k are rational numbers, both dependent upon b. In all such cases, we therefore have from (12) the generators:

$$(19)\qquad \begin{aligned} \frac{\sum_k \epsilon_k \cos bw(1-4y_k)}{\cos bw} &= \sum_{n=0}^{\infty} w^{2n} \frac{c_{b,n}}{(2n)!} , \\ \frac{\sum_k \epsilon_k \sin bw(1-4y_k)}{\cos bw} &= \sum_{n=1}^{\infty} w^{2n-1} \frac{d_{b,n}}{(2n-1)!} , \end{aligned}$$

where we have put $v = bw$. Equating powers of w gives the recurrences:

$$(20)\qquad \begin{aligned} (-1)^n \sum_k \epsilon_k [b(1-4y_k)]^{2n} &= \sum_{i=0}^{n} c_{b,n-i}(-b^2)^i \binom{2n}{2i} , \\ (-1)^{n-1} \sum_k \epsilon_k [b(1-4y_k)]^{2n-1} &= \sum_{i=0}^{n-1} d_{b,n-i}(-b^2)^i \binom{2n-1}{2i} , \end{aligned}$$

where the rightmost symbols are the binomial coefficients. Let us abbreviate

$$(21)\qquad \begin{aligned} \sum_{i=0}^{n} c_{b,n-i}(-b^2)^i \binom{2n}{2i} &= \mathfrak{C}_{b,n} , \\ \sum_{i=0}^{n-1} d_{b,n-i}(-b^2)^i \binom{2n-1}{2i} &= \mathfrak{D}_{b,n} , \end{aligned}$$

and note that the coefficient of $c_{b,n}$ ($d_{b,n}$) in these linear combinations is always 1.

Inserting now the appropriate values of ϵ_k and y_k from [1], we have the recurrences

$$(22)\qquad \begin{aligned} \mathfrak{C}_{b,n} &= (-1)^n \sum_{k=1}^{(b-1)/2} \left(\frac{k}{b}\right)[b-4k]^{2n} && \text{if } b \equiv 3 \pmod 4 , \\ \mathfrak{C}_{b,n} &= (-1)^n \sum_{2k+1<b} \left(\frac{-b}{2k+1}\right)[b-(2k+1)]^{2n} && \text{if } b \not\equiv 3 \pmod 4 , \\ \mathfrak{D}_{b,n} &= (-1)^{n-1} \sum_{k=1}^{(b-1)/2} \left(\frac{k}{b}\right)[b-4k]^{2n-1} && \text{if } b \equiv 1 \pmod 4 , \\ \mathfrak{D}_{b,n} &= (-1)^{n-1} \sum_{2k+1<b} \left(\frac{b}{2k+1}\right)[b-(2k+1)]^{2n-1} && \text{if } b \not\equiv 1 \pmod 4 . \end{aligned}$$

As examples, let us list:

(23)
$$\begin{aligned}
\mathfrak{C}_{2,n} &= (-1)^n ,\\
\mathfrak{C}_{3,n} &= (-1)^n ,\\
\mathfrak{C}_{5,n} &= (-1)^n[4^{2n} + 2^{2n}] ,\\
\mathfrak{C}_{6,n} &= (-1)^n[5^{2n} + 1^{2n}] ,\\
\mathfrak{C}_{7,n} &= (-1)^n[3^{2n} + 1^{2n} - 5^{2n}] ,\\
\mathfrak{C}_{10,n} &= (-1)^n[9^{2n} - 7^{2n} + 3^{2n} + 1^{2n}] ,\\
\mathfrak{D}_{2,n} &= (-1)^{n-1} ,\\
\mathfrak{D}_{3,n} &= (-1)^{n-1}2^{2n-1} ,\\
\mathfrak{D}_{5,n} &= (-1)^{n-1}[1^{2n-1} + 3^{2n-1}] ,\\
\mathfrak{D}_{6,n} &= (-1)^{n-1}[5^{2n-1} + 1^{2n-1}] ,\\
\mathfrak{D}_{7,n} &= (-1)^{n-1}[6^{2n-1} + 4^{2n-1} - 2^{2n-1}] ,\\
\mathfrak{D}_{10,n} &= (-1)^{n-1}[9^{2n-1} + 7^{2n-1} - 3^{2n-1} + 1^{2n-1}] .
\end{aligned}$$

By such relatively simple recurrences we express $c_{b,n}$ ($d_{b,n}$) as a linear combination of the $c_{b,m}$ ($d_{b,m}$) with $m < n$, and since $c_{b,0}$ and $d_{b,1}$ are clearly integers, so are all of these numbers integers.

Further, for $b = 1$, we have the well-known recurrences for the secant and tangent numbers, cf. [3]:

(24) $$\mathfrak{C}_{1,n} = 0, \qquad \mathfrak{D}_{1,n} = (-1)^{n-1} , \qquad (n \geqq 1)$$

and our Eqs. (22) are merely the appropriate generalization of these.

3. Comments. We have shown that the $c_{a,n}$ and $b_{a,n}$ are integers, and we have shown how they may be computed. We do not wish here to develop an elaborate theory of these numbers, and will merely close with a few brief remarks.

A. Some authors have used a notation in which the secant and tangent number coalesce into a single series, thus:

(25) $$c_{1,n} = E_n = A_{2n} , \qquad d_{1,n} = T_n = A_{2n-1} .$$

We note, from (23), that a similar joining of

$$c_{2,n} \text{ and } d_{2,n}$$

or

$$c_{6,n} \text{ and } d_{6,n}$$

is possible, because their recurrences fit together smoothly. But, in general, say, $a = 3, 5, 7$, etc., the $c_{a,n}$ and $d_{a,n}$ obey quite different laws, and therefore it does not seem desirable to attempt a joining of the complete $c_{a,n}$ and $d_{a,n}$ arrays.

B. It is clear that properties of these numbers (mod m) may be attacked fairly generally through their recurrences (22). In a less systematic way such studies have been initiated by Glaisher [4].

C. Finally, we note that recently D. J. Newman and W. Weissblum [5] have given a combinatorial interpretation of the A_n in

(26) $$\sec t + \tan t = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!} ,$$

where the notation here agrees with (25). They assert that A_n is the number of "up-down" permutations of $1, 2, \cdots, n$. Thus $A_4 = c_{1,2} = 5$ because

2143, 3142, 3241, 4132, and 4231

are the five ways in which 1234 may be permuted in which successive differences are alternatingly positive and negative. Presumably, reversals are not counted, e.g., 3412. This raises the question whether all of the $c_{a,n}$ and $d_{a,n}$ may not have some combinatorial interpretation.

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2. Milton Abramowitz & I. A. Stegun (Editors), *Handbook of Mathematical Functions*, National Bureau of Standards Applied Math. Series, 55, U. S. Government Printing, Office Washington, D. C., 1964, pp. 804–805. MR **29** #4914.

3. J. Peters, *Ten-Place Logarithm Table*, Vol. 1, Ungar, New York, 1957, Appendix, p. ix. (See Tafel 9a and 9b for T_n and E_n to $n = 30$.)

4. J. W. L. Glaisher, "The Bernoullian function," *Quart. J. Pure Appl. Math.*, v. 29, 1898, pp. 1–168.

5. D. J. Newman & W. Weissblum, "Problem 67–5," *SIAM Rev.*, v. 9, 1967, p. 121.