The Asymptotic Representation of a Class of G-Functions for Large Parameter*

By Jet Wimp

I. Introduction. In this paper, we determine the asymptotic representation for a class of G-functions,

$$(1) \qquad \Phi_n^{(M)}(\lambda) = G_{P+3,Q+2}^{M+2,1}\left(\lambda \middle| \begin{matrix} 1, a_P, 1-n, n+\gamma+1 \\ 1, \beta+1, b_Q \end{matrix}\right),$$

as $n \to \infty$, n an integer. For a treatment of the G-function, see [1].

These functions occur as coefficients in the Jacobi polynomial expansions of certain higher transcendental functions. For example, we have [2]

$$(2) \qquad f(\lambda x) = G_{P+1,Q}^{M,1}\left(\lambda x \middle| \begin{matrix} 1, a_P \\ b_Q \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(2n+\gamma)\Gamma(n+\gamma)}{\Gamma(n+\beta+1)} \Phi_n^{(M)}(\lambda) R_n^{(\alpha,\beta)}(1/x), \quad 1 < x < \infty,$$

where $\gamma = \alpha + \beta + 1$ and $R_n^{(\alpha,\beta)}(y) = P_n^{(\alpha,\beta)}(2y - 1)$ is the shifted Jacobi polynomial. Eq. (2) is valid for

$$(3) \qquad M > \frac{P+Q-1}{2}, \ |\arg \lambda| < \pi\left[M + \frac{1-P-Q}{2}\right], \qquad \lambda \neq 0,$$

and for α, β, a_i, b_j suitably restricted. (Our analysis will reveal that many of these restrictions may be dropped.)

Since $f(\lambda x)$ has an asymptotic representation in descending powers of λx, (2) may be interpreted as a summation process which converts the generally divergent expansion into a convergent one. Important special cases of (2) yield expansions for the confluent hypergeometric function $\psi(a, c; \lambda x)$ and Lommel functions.

We will treat only the case $Q - P - 1 > 0$ since the case $P + 1 \geqq Q$ may be handled by an elementary analysis. In the former, $\Phi_n^{(M)}$, as we shall see, has the unusual behavior of exponential decay as $n \to \infty$, in contrast to the latter case, where $\Phi_n^{(M)}$ behaves as inverse powers of $n!$, or at worst ($P + 1 = Q$), algebraically in n.

In Section II, we first prove three lemmas; the first establishes an integral representation for $\Phi_n^{(Q)}(\lambda)$, the second estimates for large n a closely related integral, and the third gives the desired asymptotic formula for $\Phi_n^{(Q)}(\lambda)$. Our main theorem follows when we find we can express $\Phi_n^{(M)}(\lambda)$ as a linear combination of the functions $\Phi_n^{(Q)}[\lambda \exp(\pi i(Q - M - 2k))]$.

Section III is devoted to examples.

There are quantities and assumptions about them which occur frequently in this paper, and they will always be as below:

Received April 18, 1967.

* This work was supported by the Aeronautical Research Laboratories, Office of Aerospace Research, United States Air Force, on Contract AF 33(615)-2764.

(4)
(i) M, P, Q integers $\geqq 0$, $\quad 0 \leqq M \leqq Q$;

(ii) β, γ, a_i, b_j complex constants, $\quad i = 1, 2, \cdots, P$; $\quad j = 1, 2, \cdots, Q$;

(iii) λ a complex variable $\neq 0$;

(iv) $b_j, \beta + 1 \neq 0, -1, -2, \cdots$, $\quad j = 1, 2, \cdots, Q$;

(v) $Q - P - 1 = 1/s > 0$, $\quad \omega = (s + 1/2)^{-1}$, $\quad \delta = \min\,[\omega s, \omega/2]$;

(vi) $\Delta = \sum_{j=1}^{Q} b_j - \sum_{j=1}^{P} a_j$;

(vii) $$V_n(x) = \exp\left\{-n\left[\left(\frac{2s+1}{s}\right)x^{-\omega/2} - \frac{1}{3}x^{-3\omega/2} + \frac{(3s-1)}{10(2s+1)}x^{-5\omega/2} + \cdots\right]\right\}.$$

Assumptions (iii)–(iv) are necessary to guarantee the existence of (1). We also frequently use the shorthand notation

$$\Gamma(a_P - z) = \prod_{j=1}^{P} \Gamma(a_j - z), \qquad \Gamma(b_Q - z) = \prod_{j=1}^{Q} \Gamma(b_j - z). \tag{5}$$

II. Results.

LEMMA 1. *Let*

$$-\pi/2s < \arg\lambda + \phi < \pi/2s, \qquad -\pi < \phi < \pi, \tag{6}$$

and n be a positive integer, $n > \mathrm{Re}(-\beta - 1)$, $n > -\mathrm{Re}(b_j)$, $j = 1, 2, \cdots, Q$.
Then

$$\Phi_n^{(Q)}(\lambda) = (-)^n \pi^{1/2} \int_0^{\infty \exp(i\phi)} \frac{t^{n-1} R(\lambda t)\, dt}{(1 + (1+t)^{1/2})^{2n+\gamma}(1+t)^{1/2}}, \tag{7}$$

where

$$R(\lambda t) = G_{P+2,Q+1}^{Q+1,0}\left(\lambda t \,\middle|\, \begin{matrix} a_P, (\gamma+1)/2, (\gamma+2)/2 \\ \beta+1, b_Q \end{matrix}\right). \tag{8}$$

Proof. The above result for $\phi = 0$ follows from an integral given by Saxena [3]. We give the proof, however, since it is short and since it involves relationships which will be used subsequently. We first demonstrate the convergence of the integral. At the origin, the integral is a sum of functions [4] which behave like

$$O[t^{n-1+b_j}], \qquad j = 1, 2, \cdots, Q, \quad \text{or} \quad O[t^{n+\beta}].$$

For $|t| \to \infty$, we have [5]

$$R(\lambda t) = s^{1/2}(2\pi)^{(1-s)/2s}(\lambda t)^{\Theta} \exp(-(\lambda t)^s/s)[1 + O(t^{-s})],$$

(9) $\quad |\arg\lambda + \phi| < \pi(1/s + \epsilon)$,

$$\epsilon = \begin{cases} \frac{1}{2}, & Q = P + 2, \\ 1, & Q \geqq P + 3, \end{cases} \qquad \Theta = -\tfrac{1}{2} + s(\beta - \gamma + \Delta),$$

and thus R decays exponentially, since (6) implies that $|\arg(\lambda t)^s| < \pi/2$.

Now R may be represented as the Mellin-Barnes integral [1, p. 207].

$$(10) \qquad R(\lambda t) = \frac{1}{2\pi i}\int_L \frac{\Gamma(\beta+1-z)\Gamma(b_Q-z)(\lambda t)^z dz}{\Gamma(a_P-z)\Gamma((\gamma+1)/2-z)\Gamma((\gamma+2)/2-z)},$$

where L runs from $-i\infty$ to $i\infty$ in such a manner that all the poles of the integrand are to the right of the path; L can be chosen so that the integral converges absolutely.

When (10) is substituted in (7), the order of integration can be interchanged by absolute convergence, and Lemma 1 follows by application of the known integral

$$(11) \quad \int_0^{\infty \exp(i\phi)} \frac{t^a(1+(1+t)^{1/2})^b}{(1+t)^{1/2}}\,dt = 2^{2a+b+2}\,\frac{\Gamma(a+1)\Gamma(-2a-b-1)}{\Gamma(-a-b)},$$

for $-\pi < \phi < \pi$, Re $(a + b/2) < -1/2$, Re $(a) > -1$.

LEMMA 2. *Let*

$$(12) \qquad L_n(z) = \int_0^{\infty \exp(i\phi)} \frac{t^{n+c_1}e^{-zt^s}}{[1+(1+t)^{1/2}]^{2n+c_2}(1+t)^{1/2}}\,[1+M(t)t^{-s}]dt,$$

where c_1, c_2, z are all complex quantities, n and s are real, $0 < s \leqq 1$, $M(t)$ is analytic and bounded in the sector $|\arg t| \leqq |\phi| < \pi$, $|t| \geqq \epsilon > 0$, and for n sufficiently large, $t^n M(t)$ is finite as $t \to 0$ in that sector. Also

$$(13) \qquad -\pi(s+\tfrac{1}{2}) < \arg z < \pi(s+\tfrac{1}{2}), \qquad z \neq 0,$$

and $\phi = \arg t_0$, where t_0 is the root of

$$(14) \qquad t_0^s(1+t_0)^{1/2} = n/zs = \rho,$$

which for large n behaves as

$$(15) \qquad t_0 \sim \rho^\omega.$$

Then the integral converges for n sufficiently large and we have

$$(16) \qquad L_n(z) = (2\pi\omega/n)^{1/2}\rho^{\omega(c_1-c_2/2+3/4)}V_n(\rho)[1+O(n^{-\delta})], \qquad n \to \infty.$$

Proof. The proof is by application of the method of steepest descent. First, let ϕ be unrestricted. Then the integral will converge if

$$(17) \qquad -\pi < \phi < \pi, \qquad -\pi/2 < \arg z + s\phi < \pi/2,$$

but since $\phi \sim -\omega \arg z$, $n \to \infty$, the conditions (17) reduce to (13) and, indeed, $|\phi| < \pi$, as stated in the hypotheses.

Let

$$(18) \qquad \begin{aligned} L_n(z) &= \int_0^{\infty \exp(i\phi)} G(t)e^{A_n(t)}dt, \\ A_n(t) &= -zt^s + n\ln t - 2n\ln[1+(1+t)^{1/2}]. \end{aligned}$$

The saddle point in question is at $A_n^{(1)}(t_0) = 0$, which gives (14), and one verifies that

$$(19) \qquad t_0 = \rho^\omega\left[1 - \frac{1}{(2s+1)}\rho^{-\omega} + \frac{s}{(2s+1)^2}\rho^{-2\omega} + \cdots\right].$$

For higher derivatives of A_n we have

$$A_n^{(k)}(t_0) = O(nt_0^{-k-1/2}), \qquad k \geqq 1. \tag{20}$$

If we write

$$S_n(z) = t_0 e^{A_n(t_0)} \int_{1-\mu_n}^{1+\mu_n} G(t_0 u) e^{\nu\phi(u)} du, \tag{21}$$

where

$$\begin{gathered} \mu_n = \delta_n/|t_0|, \qquad \nu = (-t_0^2/2)A_n^{(2)}(t_0), \\ \phi(u) = -(u-1)^2 - \frac{t_0 A_n^{(3)}(t_0)(u-1)^3}{3A_n^{(2)}(t_0)} - \frac{t_0^2 A_n^{(4)}(t_0)(u-1)^4}{12A_n^{(2)}(t_0)} \cdots, \end{gathered} \tag{22}$$

and make the choice

$$\delta_n = n^{\omega(1-s/3)}, \tag{23}$$

then $L_n(z) \sim S_n(z)$, the neglected portion of the integrand being exponentially subdominant to $S_n(z)$: in fact, we have

$$\frac{\exp[A_n(t_0 \pm \delta_n e^{i\phi})]}{\exp[A_n(t_0)]} = \exp[-Kn^{\omega s/3} \exp((i\omega/2)\arg z) + O(1)], \tag{24}$$

where $K > 0$. By (20), every term in the series for $\phi(u)$ is a bounded function of n. Thus we can make the substitution $\phi = -\zeta^2$ in (21) and proceed in the usual fashion. The lemma follows, since $\exp[A_n(t_0)] = V_n(\rho)$, and $\nu^{-1/2}$, $G(t_0)$ are easily estimated.

LEMMA 3. *Let* $|\arg \lambda| < \pi(1/2s + 1)$ *and* n *be integral. Then*

$$\Phi_n^{(Q)}(\lambda) = (-)^n(\pi\omega s)^{1/2}(2\pi)^{1/2s}\lambda^{d_1}n^{d_2}V_n[n\lambda^{-s}][1 + O(n^{-\delta})], \tag{25}$$

$n \to \infty$, *where*

$$d_1 = \frac{\omega}{2}\left[-\frac{1}{2} + s\left(\beta + \Delta + \frac{1}{2}\right)\right], d_2 = \omega\left[-\frac{\gamma}{2} - 1 + s\left(-\frac{1}{2} - \gamma + \beta + \Delta\right)\right]. \tag{26}$$

Proof. In the expansion (9), include the remainder term, i.e., write $[1 + O(t^{-s})] \equiv [1 + M(t)t^{-s}]$ and substitute (9) in (7). Then the hypotheses of Lemma 2 are satisfied, and a straightforward identification of parameters gives (25).

Now, using the reflection formula for the gamma function, we find that

$$\begin{aligned} \Phi_n^{(M)}(\lambda) = \frac{\pi^{M-Q}}{2\pi i} \int_L & \frac{\Gamma(1-z)\Gamma(\beta+1-z)\Gamma(b_Q - z)\Gamma(z)}{\Gamma(a_P - z)\Gamma(1-n-z)\Gamma(n+\gamma+1-z)} \\ & \times \prod_{j=M+1}^{Q} \sin \pi(b_j - z)\lambda^z dz. \end{aligned} \tag{27}$$

By decomposing the sine product into its exponential factors, we may write

$$\Phi_n^{(M)}(\lambda) = (-2\pi i)^{M-Q} \exp\left(-\pi i \sum_{j=M+1}^{Q} b_j\right) \sum_{k=0}^{Q-M} \mu_k^{(M)} \Phi_n^{(Q)}[\lambda e^{\pi i(Q-M-2k)}], \tag{28}$$

where $\mu_k^{(M)}$ is the coefficient of x^k in the product

$$\prod_{j=M+1}^{Q} (1 - x \exp(2\pi i b_j)) , \tag{29}$$

and empty sums are zero, empty products unity.

We thus have our main

THEOREM. *Let* $M > (P + Q - 1)/2$, $|\arg \lambda| < \pi[1/2s + 1 + M - Q]$, *n integral. Then*

$$\begin{aligned} \Phi_n^{(M)}(\lambda) = {} & (-)^n (\pi\omega s)^{1/2} (2\pi)^{1/2s} \lambda^{d_1} n^{d_2} (-2\pi i)^{M-Q} \\ & \times \exp\left(-\pi i\left[\sum_{j=M+1}^{Q} b_j - d_1(Q - M)\right]\right) \\ & \times \sum_{k=0}^{Q-M} \mu_k^{(M)} \exp(-2\pi i d_1 k) V_n[n(\lambda e^{\pi i(Q-M-2k)})^{-s}] \\ & \times [1 + O(n^{-\delta})] , \end{aligned} \tag{30}$$

$n \longrightarrow \infty$, *where* d_1, d_2, $\mu_k^{(M)}$ *are as above.*

III. Examples. An expansion given in [2] is

$$(\lambda x)^a \Psi(a, c; \lambda x) = \sum_{n=0}^{\infty} C_n(\lambda) T_n^*(1/x) , \tag{31}$$

$1 \leqq x \leqq \infty$, $a + 1 - c, a \neq 0, -1, -2, \cdots$, $|\arg \lambda| < 3\pi/2$, $\lambda \neq 0$, where

$$C_n(\lambda) = \frac{\epsilon_n}{\pi^{1/2}\Gamma(a)\Gamma(a+1-c)} G_{3,4}^{4,1}\left(\lambda \middle| \begin{matrix} 1, 1-n, n+1 \\ 1, \frac{1}{2}, a, a+1-c \end{matrix}\right), \tag{32}$$

$\epsilon_0 = 1$, $\epsilon_n = 2$, $n > 0$ and $T_n^*(y) = T_n(2y - 1)$ is the shifted Chebyshev polynomial. Our theorem gives

$$\begin{aligned} & C_n(\lambda) = \sigma_1(\lambda)(-)^n n^r \exp(-3n^{2/3}\lambda^{1/3})[1 + O(n^{-1/3})] , \\ & |\arg \lambda| < \frac{3\pi}{2}, \qquad \sigma_1(\lambda) = \frac{4\pi^{1/2}\lambda^{(4a-2c+1)/6}}{3^{1/2}\Gamma(a)\Gamma(a+1-c)} e^{\lambda/3}, \qquad r = \frac{2}{3}(2a - c - 1) . \end{aligned} \tag{33}$$

This provides the multiplicative constant missing in Németh's result [6] and agrees with a result in Miller [7].

As shown in [2], $C_n(\lambda)$ satisfies a homogeneous third-order difference equation. The Birkhoff-Trjitzinsky theory of singular difference equations [8] asserts that there exist two other linearly independent solutions of the difference equation which have the behavior

$$\begin{aligned} \phi_{2,n}(\lambda) &= \sigma_2(\lambda)(-)^n n^r \exp(-3\omega_1 n^{2/3}\lambda^{1/3})[1 + O(n^{-1/3})] , \\ \varphi_{3,n}(\lambda) &= \sigma_3(\lambda)(-)^n n^r \exp(-3\omega_2 n^{2/3}\lambda^{1/3})[1 + O(n^{-1/3})] , \end{aligned} \tag{34}$$

where $\omega_1 = (-1 + 3^{1/2}i)/2$, $\omega_2 = \bar{\omega}_1$, $|\arg \lambda| < 3\pi/2$.

In [2], it was stated that, for $|\arg \lambda| < \pi$, $C_n(\lambda)$ could be computed by using the difference equation in the backward direction, but the proof given there is incorrect. Nevertheless, the statement is true, as is easily seen by using (33)–(34). Let $\phi_{1,n}(\lambda) = C_n(\lambda)$. In the notation of [2], we find

$$(35)\qquad \begin{aligned} \tau_\nu &= O\{\nu^{\eta_1} \exp [3\nu^{2/3}\lambda^{1/3}]\} , \\ \gamma_\nu &= O\{\nu^{\eta_2} \exp [-3\nu^{2/3}\lambda^{1/3}(2 + \omega_2)]\} , \\ \delta_\nu &= O\{\nu^{\eta_3} \exp [-3\nu^{2/3}\lambda^{1/3}(2 + \omega_1)]\} . \end{aligned}$$

But

$$(2 + \omega_2) = \overline{(2 + \omega_1)} = 3^{1/2}e^{\pi i/6},$$

and the requirement that $\gamma_\nu \to 0, \delta_\nu \to 0$ as $\nu \to \infty$ reduces to $|\arg (\exp (\pm\pi i/6)\lambda^{1/3})| < \pi/2$, or $|\arg \lambda| < \pi$.

For a Lommel function, we have

$$(36)\qquad \begin{aligned} &(\lambda x)^{1-a}2^{2-2a}\Gamma(1 - a - b)\Gamma(1 - a + b)S_{2a-1,2b}(2(\lambda x)^{1/2}) \\ &\qquad = \sum_{n=0}^{\infty} D_n(\lambda)T_n^*(1/x) , \qquad 1 \leqq x \leqq \infty , \end{aligned}$$

and by [9, formula (32)], (2) and the theorem, we find

$$(37)\qquad D_n(\lambda) = (-)^n 2^{3/2}\pi\lambda^{(1-a)/2}n^{-a} \exp [-4n^{1/2}\lambda^{1/4}][1 + O(n^{-1/2})] ,$$

$|\arg \lambda| < 2\pi$.

Also by [9, formula (26)], (2) and the theorem, we get

$$(38)\qquad (2\pi\lambda x)^{1/2}I_a[(\lambda x)^{1/2}]K_a[(\lambda x)^{1/2}] = \sum_{n=0}^{\infty} E_n(\lambda)T_n^*\left(\frac{1}{x}\right),$$

$1 \leqq x \leqq \infty$, and

$$(39)\qquad \begin{aligned} E_n(\lambda) &= (-)^n 2^{3/2}\lambda^{1/8}n^{-3/4} \exp [-2\cdot 2^{1/2}n^{1/2}\lambda^{1/4}] \\ &\quad \times \sin \left[\pi\left(\frac{3}{8} - a\right) + 2\cdot 2^{1/2}n^{1/2}\lambda^{1/4}\right][1 + O(n^{-1/2})] , \qquad |\arg \lambda| < \pi , \end{aligned}$$

where $I_a(z)$ and $K_a(z)$ are the modified Bessel functions.

There are a number of examples in the literature of expansions of the kind

$$(40)\qquad f(\lambda x) = \sum_{n=0}^{\infty} F_n(\lambda)T_n\left(\frac{1}{x}\right), \qquad |x| \geqq 1 ,$$

where Lemma 2 alone suffices to obtain an asymptotic formula for the coefficients, since F_n has an appropriate integral representation, see [10], [11], [7].

For instance

$$(41)\qquad \begin{aligned} F_n(\lambda) &= (2\lambda)^{1/2}\pi^{-3/2} \int_0^\infty \frac{e^{-\lambda x}K_0(\lambda x)x^{n-1/2}dx}{(1 + x^2)^{1/2}[1 + (1 + x^2)^{1/2}]^n} \\ &= \frac{1}{2\pi}\int_0^\infty \frac{\exp (-2\lambda t^{1/2})t^{n/2-1}}{(1 + t)^{1/2}[1 + (1 + t)^{1/2}]^n} [1 + M(t)t^{-1/2}]dt , \end{aligned}$$

by the asymptotic formula for K_0, and thus

$$(42)\qquad F_n(\lambda) = \frac{(2\lambda)^{1/4}}{\pi^{1/2}} n^{-3/4} \exp [-2(2\lambda n)^{1/2}][1 + O(n^{-1/2})] .$$

Formula (42) is valid not only for values of λ for which the integrals (41) converge, but also furnishes a valid asymptotic representation for the analytic continuation of $F_n(\lambda)$ into the sector $|\arg \lambda| < \pi$. It agrees with the result given (for λ real) in [10a]; there $F_n(\lambda)$ occurs in the Chebyshev polynomial expansion for one of the asymptotic components of the Bessel function $J_0(x)$.

IV. Conclusions. Actually, the expansion in [2] is more general, with coefficients

$$G_{P+2,Q+2}^{M+2,K}\left(\lambda\middle|\begin{matrix}a_P, 1-n, n+\gamma+1\\ 1, \beta+1, b_Q\end{matrix}\right) \tag{43}$$

and

$$f(\lambda x) = G_{P,Q}^{M,K}\left(\lambda x\middle|\begin{matrix}a_P\\ b_Q\end{matrix}\right) \tag{44}$$

where a_1 and K are no longer necessarily 1. Of course, one can always make $a_1 = 1$ by expanding $(\lambda x)^{1-a_1}f(\lambda x)$ instead of $f(\lambda x)$, but when $K > 1$ the situation changes considerably, and the analysis is more complicated. Preliminary work indicates that the asymptotic representation of (43) is then contaminated with terms which are purely algebraic in n, unless $a_1 = a_2 = \cdots = a_K = 1$; thus the convergence of the more general expansion is greatly inferior to the convergence of (2). In fact, for (2), our analysis enables us to weaken certain hypotheses: we can conclude that the expansion converges at the end points $x = 1$, $x = \infty$, and that various restrictions on a_i, b_j can be dropped, see [2]. The expansion is now seen to converge for $0 \leqq M \leqq Q$, $Q > P + 1$, b_j, $\beta + 1$ not zero or negative integers, $M > \frac{1}{2}(P + Q - 1)$, $|\arg \lambda| < \pi(M + (1 - P - Q)/2)$, $\lambda \neq 0$, $1 \leqq x \leqq \infty$. The same applies to the expansions in [10], [11], where λ was assumed real. Lemma 2 shows that those expansions converge for $|\arg \lambda| < \pi$.

The advantage of our approach over using the Langer method [12] on the differential equation satisfied by $\Phi_n^{(M)}(\lambda)$ is twofold: the Langer method, at the present time, is only a formal procedure, and it does not give the leading constant. (However, if higher terms in the expansion (30) are needed, they are most easily obtained from the differential equation.)

Since $\Phi_n^{(M)}(\lambda)$ satisfies a difference equation in n of order max $[Q + 1, P + 2]$, it might be thought that the subnormal solutions encountered in the Birkhoff-Trjitzinsky theory could be used to prove the theorem. But the problems involved in this approach are rather serious. Not only is the lead constant undetermined but, for general M, P, Q, the application of the theory (even in its formulation by Turrittin [13]) presents insuperable algebraic difficulties.

V. Acknowledgements. The author expresses his thanks to Professor Arthur Erdélyi, the Mathematical Institute, Edinburgh, for his suggestions, and also to his colleague, Yudell Luke, for a critical reading of the manuscript.

The Mathematical Institute
The University
Edinburgh, Scotland

1. A. Erdélyi, et al., *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, 1953, Chapter 5. MR **15**, 419.

2. Y. L. LUKE & JET WIMP, "Jacobi polynomial expansions of a generalized hypergeometric function over a semi-infinite ray," *Math. Comp.*, v. 17, 1963, pp. 395–404. MR **28** #255.

3. R. K. SAXENA, "Some formulae for the G-function," *Proc. Cambridge Philos. Soc.*, v. 59, 1963, pp. 347–350. MR **26** #6448.

4. Reference 1, p. 208, formula (5).

5. C. S. MEIJER, "On the G-function. I," *Nederl. Akad. Wetensch.*, v. 49, 1946, p. 234, formulae (22)–(24). MR **8**, 156.

6. GÉZA NÉMETH, "Polynomial approximations to the function $\psi(a,c;x)$," Reports of the Central Institute for Physics, Budapest, 1965.

7. G. F. MILLER, "On the convergence of the Chebyshev series for functions possessing a singularity in the range of representation," *J. Soc. Indust. Appl. Math. Ser. Numer. Anal.*, v. 3, 1966, pp. 390–409.

8. GEORGE BIRKHOFF & W. J. TRJITZINSKY, "The analytic theory of singular difference equations," *Acta Math.*, v. 60, 1932, pp. 1–89.

9. Reference 1, p. 218.

10. GÉZA NÉMETH, (a) "Chebyshev expansions of Bessel functions. I: $J_\nu(x)$ and $N_\nu(x)$"; (b) "Chebyshev expansions of integral sine and integral cosine functions," same reports as Reference 6.

11. GÉZA NÉMETH, "Chebyshev expansions for Fresnel integrals," *Numer. Math.*, v. 7, 1965, pp. 310–312. MR **32** #3265.

12. R. E. LANGER, "Turning points in linear asymptotic theory," *Bol. Soc. Mat. Mexicana* (2), v. 5, 1960, pp. 1–12. MR **22** #11183.

13. H. L. TURRITTIN, "The formal theory of systems of irregular homogeneous linear difference and differential equations," *Bol. Soc. Mat. Mexicana*, (2), v. 5, 1960, pp. 255–264. MR **25** #349.