TABLE 2

Solution of One System Equations

Array \ Step	0	1	2	3
1	$A \mid b_1$	$P = A + B \mid c_1 = b_1 + b_2$	Solve $Py_1 = c_1$	$x_1 = .5(y_1 + y_2)$
2	$B \mid b_2$	$Q = P - 2B \mid c_2 = c_1 - 2b_2$	Solve $Qy_2 = c_2$	$x_2 = x_1 - y_2$

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A Note on the Effect of Conditionally Stable Correctors

By Fred T. Krogh

We say a corrector of the form

$$y_{n+1} = \sum_{i=0}^{k-1} A_i y_{n-i} + h \sum_{i=-1}^{k-1} a_i y'_{n-i} \tag{1}$$

is *conditionally stable* if the polynomial

$$\rho(z) = z^k - \sum_{i=0}^{k-1} A_i z^{k-1-i} \tag{2}$$

has all of its roots in the unit disk, roots of unit magnitude are simple, and there is at least one root of unit magnitude besides the root $z = 1$ (which must be a root since it is assumed that Eq. (1) is satisfied if y is a constant). In [1], Stetter obtains the remarkable result that a predictor-corrector algorithm using Simpson's rule (a conditionally stable corrector) will be relatively stable* for sufficiently small h provided the predictor is chosen judiciously and the corrector is only applied once. However, his result applies only to the integration of a single differential equation. It is the purpose of this note to point out that no result of this type can

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* From the bewildering array of words in the current literature which describe stability, those used here seem to the author to be most descriptive. Different words are used in [1]. Several definitions of "relatively stable" are given below. In practice they are essentially equivalent.

be obtained if one considers systems of differential equations. (This is contrary to that stated just above Eq. (2.25) in [2]. The analysis there is not based on the linear equation $y' = \lambda y$ as it is in [1] and here.)

The stability characteristics of an algorithm for solving systems of differential equations can be obtained by examining its stability for single equations of the form

$$y' = \lambda y \tag{3}$$

where λ is a complex constant (see references [3] and [4]). The stability of the algorithm in solving (3) depends in turn on the behavior of the roots to the characteristic equation

$$p(s, z) = 0 \tag{4}$$

where p is a polynomial in s and z, and $s = h\lambda$. It is well known, see for example Chapter I of [5], that except for branch points the roots $r_1, r_2, \cdots$ of Eq. (4) are analytic functions of s. It is easy to show that the roots of $p(0, z)$ coincide with those of $\rho(z)$. Assuming the corrector is conditionally stable, we label the roots of p such that $r_1(0) = 1$ and $|r_2(0)| = 1$.

If $p(s, z)$ is a polynomial in z^j (j an integer greater than one), then the method is only using every jth point. The error propagation of such a method should be studied by replacing s with s/j and z^j with z in Eq. (4). We assume that the characteristic equation has been so formulated.

THEOREM. *If $p(s, z)$ is irreducible, then for sufficiently small ϵ (>0), there is a number s with modulus ϵ such that $|r_2(s)| > |r_1(s)|$.*

Proof. Since roots of unit magnitude are simple at $s = 0$, it follows that $\phi(s) = r_2(s)/r_1(s)$ is analytic in a neighborhood of the origin. A simple application of the maximum principle, see for example [6, p. 165], to $\phi(s)$ reveals that for ϵ sufficiently small there is a number s with modulus ϵ such that $|r_2(s)| > |r_1(s)|$, provided $r_1(s) \not\equiv cr_2(s)$, where c is a constant.

If $r_1(s) \equiv cr_2(s)$,† $p(s, z) = 0$ and $p(s, cz) = 0$ both have $z = r_1(s)$ as a root in a neighborhood of the origin. Since p is irreducible, this implies the two equations define the same algebraic function (cf. [5]). This in turn implies that c is a jth root of one ($j > 1$) and $p(s, z)$ is a polynomial in z^j. Thus $r_1(s) \not\equiv cr_2(s)$ and the theorem is proved.

A method with a $p(s, z)$ which is not irreducible is of dubious value. From such a method could be derived a simpler method with the same principal root, $r_1(s)$, and without some of the extraneous roots.

Ralston [7] defines a method to be relatively stable at s provided $|r_i(s)| \leqq |r_1(s)|$, $i = 2, 3, \cdots$, and in the case of equality that r_i be simple. Stetter simply requires $|r_1(s)| \leqq |r_i(s)|$. Crane and Klopfenstein [3], and Krogh [4] use Ralston's definition with e^s substituted for $r_1(s)$. With either of the first two definitions, the above theorem shows that a method which uses a conditionally stable corrector is not relatively stable for arbitrarily small values of $|s|$. The same result for the last definition is obtained from the first paragraph of the proof with $\phi(s) = r_2(s)/e^s$.

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Midpoint Quadrature Formulas

By Seymour Haber

A family of quadrature formulas for the interval (0, 1) can be constructed in the following manner: For any positive integer n, we partition (0, 1) into subintervals $I_1, I_2, \cdots, I_n$ (I_1 being the leftmost, I_2 adjacent to it, etc.) of lengths $a_1, a_2, \cdots, a_n$, respectively. Now let x_k be the midpoint of I_k, for $k = 1, \cdots, n$, and take

$$a_1 f(x_1) + \cdots + a_n f(x_n) \tag{1}$$

as the approximation to $\int_0^1 f(x)dx$. The simplest of these rules is the "Euler's" or "midpoint" rule

$$\int_0^1 f(x)dx \approx f(\tfrac{1}{2}) .$$

We will refer to the members of this family as "midpoint quadrature formulas" and determine their properties. We first find their "degrees of precision"—that is, for any formula, the highest integer p such that the formula is exact for all polynomials of degree p or lower.

Theorem 1. *The degree of precision of a midpoint quadrature formula is* 1.

Proof. The formula is exact for constants, since necessarily $a_1 + a_2 + \cdots + a_n = 1$. To check the exactness of the formula for $f(x) = x$, we first note that

$$x_1 = \frac{a_1}{2}, x_2 = a_1 + \frac{a_2}{2}, \cdots, x_n = a_1 + \cdots + a_{n-1} + \frac{a_n}{2} . \tag{2}$$

So for the integral $\int_0^1 x\, dx$, (1) gives us

$$a_1(a_1/2) + a_2(a_1 + a_2/2) + \cdots + a_n(a_1 + \cdots + a_{n-1} + a_n/2) .$$

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