The Maxima of $P_r(n_1, n_2)$

By M. S. Cheema* and H. Gupta

1. In this note, we study the maxima of $P_r(n_1, n_2)$, the number of partitions of the vector $(n_1, n_2)$ into exactly $r$ parts (vectors) with positive integral components.

The generating function $\phi_r(x_1, x_2)$ for $P_r(n_1, n_2)$ is given by

$$\prod_{k_1, k_2=1}^{\infty} (1 - x_1^{k_1}x_2^{k_2})^{-1} = 1 + \sum_{r=1}^{\infty} z^r \phi_r(x_1, x_2)$$

$$(1.2) \phi_r(x_1, x_2) = 1 + \sum_{n_1, n_2=1}^{\infty} P_r(n_1, n_2)x_1^{n_1}x_2^{n_2}.$$  

2. If $q_r(n_1, n_2)$ denotes the number of partitions of $(n_1, n_2)$ into at most $r$ parts (vectors) with nonnegative integral components, then it follows that $q_r(n_1, n_2) = P_r(n_1 + r, n_2 + r)$. It is clear that $q_r(n_1, n_2)$ is an increasing function of $r$ for $1 \leq r < n_1 + n_2$, and becomes constant for $r \geq n_1 + n_2$, on the other hand $P_1(n_1, n_2) = 1$ and $P_r(n_1, n_2) = 0$ for $r > \min(n_1, n_2)$. From the table of values of $P_r(n_1, n_2)$ computed by Cheema, we notice that for $n_1 \geq n_2 > 0$, there is a unique $s$ such that

$$P_1(n_1, n_2) < P_2(n_1, n_2) < \cdots < P_s(n_1, n_2) \geq P_{s+1}(n_1, n_2) \geq \cdots \geq P_{n_1}(n_1, n_2).$$

We use $s$ in this sense in all that follows. The values of $s$ were computed for all $n_1, n_2 \leq 50$. We might remark that a similar conjecture holds for the number of partitions of $n$ into exactly $r$ summands. An explicit formula for $P_r(n_1, n_2)$ for general $r$ is not known, $P_r(n_1, n_2)$ do satisfy a recurrence relation and behave very much like a polynomial in $n_1, n_2$, i.e., $P_r(n_1, n_2)$ is a semipolynomial of degree $r - 1$ in $n_1$ and $n_2$ relative to modulus $r!$ as shown by Wright [2]. Thus

$$P_r(n_1, n_2) = \sum_{t_1=1}^{r} \sum_{t_2=1}^{r} \beta(t_1, t_2, n_1, n_2)n_1^{t_1-1}n_2^{t_2-1},$$

where $\beta(t_1, t_2, n_1, n_2)$ depends on $r, t_1, t_2$ and on the residues of $n_1, n_2$ to moduli 1, 2, 3, \ldots, $[r/2]$, but not otherwise on $n_1, n_2$. A rough estimate for $s$ is obtained by studying the maxima of a function which behaves very much like $P_r(n_1, n_2)$.
3. For $n_1, n_2$ large compared to $r$, $P_r(n_1, n_2)$ behaves very much like the function

$$\frac{1}{r!} \binom{n_1 - 1}{r} \binom{n_2 - 1}{r}.$$ 

Using this estimate and using $P_r(n_1, n_2) \geq P_{r+1}(n_1, n_2)$, we obtain $s = \min (r, n_1, n_2)$, where $r$ is the least positive integer satisfying

$$\frac{(n_1 - r)(n_2 - r)}{r(r + 1)} \leq r^2 (r + 1).$$ 

Roughly such an $r$ is given by $(n_1 n_2)^{1/3}$. If $n_1 = n_2 = n$, then as in [1]

$$P_r(n, n) \approx \frac{1}{r!} \left( \frac{n - 1}{r - 1} \right)^2 \exp \left( \frac{r^3 \log r}{n^2} \right).$$ 

Hence $P_r(n, n) \geq P_{r+1}(n, n)$ implies that

$$\left( n - s \right)^2 \leq (s + 1)s^{2 - 2s - 3s + 1/n^2}.$$ 

As a rough estimate we have $s \approx n^{2/3}$. The inequality (3.3) gives a good estimate for $s$ for a particular $n$. Thus for $n = 50$, the value of $s$ by (3.3) is 14, while the actual value is 13. For $n = 52$, $s = 14$ both by the inequality and the tables.

The University of Arizona
Tucson, Arizona