Explicit $O(h^2)$ Bounds on the Eigenvalues of the Half-L*

By Blair K. Swartz

0. Summary and Survey. This paper is concerned with obtaining strict upper and lower bounds on the eigenvalues of a particular nontrivial convex membrane, the half-L, which is fixed or free at the boundary. The upper bound is obtained from a matrix eigenvalue calculation; the matrix problem may be regarded as a difference scheme although it is derived using piecewise linear functions in a Rayleigh quotient. The lower bound is than calculated from the upper bound using an elementary formula. The validity of this formula is proved by extensions of Weinberger's techniques [3]. Difficulties encountered in determining similar results for the nonconvex L-shaped membrane are indicated. A numerical example illustrates the results. An appendix contains some pointwise bounds on normalized eigenfunctions.

An annotated survey of the literature concerned with estimating the eigenvalues of the Laplacian is contained in [4]; many other references are found in [10]. The use of piecewise linear functions in variational principles is discussed in [1], [2], [5]; piecewise bilinear in [6], [7], [9, pp. 331–334]. A general discussion of Rayleigh-Ritz eigenvalues is found in [8].

Much of the literature concerned with strict bounds on the eigenvalues seems to use the eigenvalues of the discrete Laplacian or a related matrix rather than the eigenvalues associated with the Rayleigh-Ritz method. The lower bounds of [19], [3], [13], and the simultaneous two-sided bounds in [10] are $O(h)$ bounds as a result of embedding a general region in a union of squares [9, p. 339], [4, p. 30]. It is not clear how the lower bound in [14] behaves as $h \to 0$. (See also [4, pp. 30–31].) Other results have been asymptotic: the discrete eigenvalue is the continuous one (in certain cases) except for a term $\gamma h^2 + o(h^2)$, γ unknown [4], [9]; a result quite useful in extrapolation to zero.

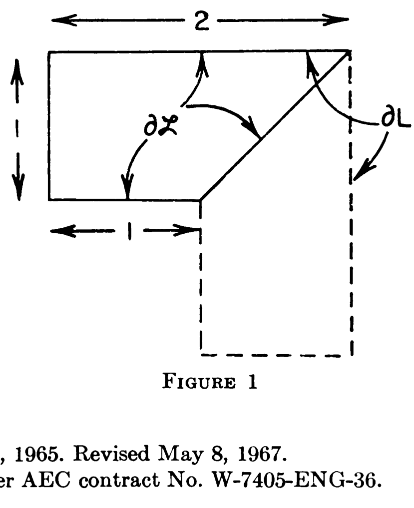


FIGURE 1

Received November 15, 1965. Revised May 8, 1967.

* Work performed under AEC contract No. W-7405-ENG-36.

1. General Approach, Results, and Notation. We shall be concerned throughout with two regions in the plane (Fig. 1).

The primary interest will be in the lowest nontrivial eigenvalue of two problems:

I. $\Delta u = -\lambda u$ in $\mathcal{L}$, $u = 0$ on $\partial\mathcal{L}$ (Sections 2 and 3, two different meshes); and

II. $\Delta u = -\lambda u$ in $\mathcal{L}$, $\partial u/\partial\nu = 0$ on $\partial\mathcal{L}$ (Section 4).

However, the bounds on λ are shown to hold for the higher eigenvalues as well (Section 5).

The general approach is a variational one. In each case

$$\lambda = \min_{W} \frac{\iint |\nabla w|^2 dA}{\iint w^2 dA}$$

where W is a class of continuous, piecewise differentiable functions satisfying the boundary conditions. The approximating procedure consists of placing a mesh on the region in a particular manner, triangulating it, and considering a class V of continuous functions, piecewise linear on the triangles, satisfying the boundary condition. If $\mathbf{x}_i$ are the interior mesh points, each $v \in V$ may be represented by a vector $\mathbf{v} = \{v(\mathbf{x}_i)\}$. We then have, with $\iint|\nabla v|^2 dA \equiv N(\mathbf{v}) \equiv (\mathbf{v}, A\mathbf{v})$, $\iint v^2 dA \equiv D(\mathbf{v}) \equiv (\mathbf{v}, B\mathbf{v})$;

$$\lambda \leqq \lambda_h = \min_{v \in V} \frac{\iint |\nabla v|^2 dA}{\iint v^2 dA} = \min_{\mathbf{v}} \frac{N(\mathbf{v})}{D(\mathbf{v})} = \min_{\mathbf{v}} \frac{(\mathbf{v}, A\mathbf{v})}{(\mathbf{v}, B\mathbf{v})} .$$

(N and D vary from section to section as the meshes and/or problem vary.) It is easily shown that A and B are symmetric positive-definite sparse matrices. ($A\mathbf{v}$, for example, will in each case be essentially $-h^2 \Delta_h \mathbf{v}$.) The numerical calculation of λ_h then amounts to solving $A\mathbf{v} = \lambda_h B\mathbf{v}$, which can be done by successive over-relaxation, as indicated in Section 7.

To obtain the lower bound on λ in terms of λ_h one shows three things. With u, the eigenfunction, normalized so that $\iint u^2 dA = 1$, define

$$\bar{u}(\mathbf{x}) = \iint_{S(\mathbf{x})} u\,dA/h^2$$

where $S(\mathbf{x})$ is a square of side h centered on $\mathbf{x}$, and set $\mathbf{u} = \{\bar{u}(\mathbf{x}_i)\}$. For Problem I it is then shown that

$$D(\mathbf{u}) \geqq h^2(\mathbf{u}, \mathbf{u}) - h^2 N(\mathbf{u})/4 , \tag{1.1}$$

$$h^2(\mathbf{u}, \mathbf{u}) \geqq 1 - h^2\lambda/\pi^2 , \quad \text{and} \tag{1.2}$$

$$N(\mathbf{u}) \leqq \lambda . \tag{1.3}$$

As an immediate consequence we have, with $c = 1/4 + 1/\pi^2$ and $ch^2\lambda < 1$,

$$\lambda \leqq \lambda_h \leqq \lambda/(1 - ch^2\lambda) , \tag{1.4}$$

indicating that $\lambda_h - \lambda \leqq O(h^2)$. Solving (1.4) for λ yields the explicit double bounds

$$\lambda_h/(1 + ch^2\lambda_h) \leqq \lambda \leqq \lambda_h . \tag{1.5}$$

Finally, we show the same result holds for the higher eigenvalues, $\lambda^{(k)}$:

$$\lambda_h^{(k)}/(1 + ch^2\lambda_h^{(k)}) \leqq \lambda^{(k)} \leqq \lambda_h^{(k)} \quad \text{if} \quad ch^2\lambda_h^{(k)} < 1 . \tag{1.6}$$

For problem II it is necessary to subtract an additional term, $h^2 u_0^2/4$, from (1.1), where u_0 is a bound on the normalized eigenfunction in certain subsets of $\mathfrak{L}$. (1.4) then becomes

$$\lambda \leqq \lambda_h \leqq \lambda/(1 - ch^2\lambda - h^2 f(\lambda, h)/4) \equiv g_h(\lambda) . \tag{1.4a}$$

(See Appendix IV, Eq. (IV.2).) Since $f(\lambda, h)$ is increasing in λ for each h, g has the same property. One then concludes, with λ the first nontrivial eigenvalue of problem II,

$$g_h^{-1}(\lambda_h) \leqq \lambda \leqq \lambda_h ; \tag{1.5a}$$

g_h is easily inverted numerically. The higher eigenvalues may be bounded below by inverting a corresponding function (5.1), given upper bounds $\lambda_h^{(1)}, \cdots, \lambda_h^{(k)}$.

We now turn to particulars.

2. $\Delta u = -\lambda u$, $u = 0$ **on** $\partial\mathfrak{L}$. For reasons to become apparent we place a mesh on the full L and triangulate it as in Fig. 2.

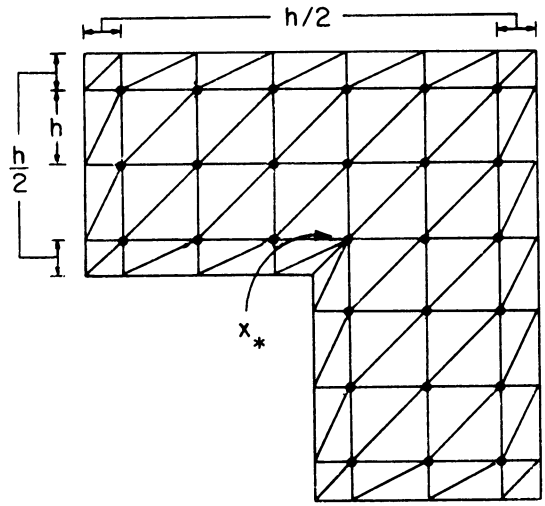


FIGURE 2

All v vanish on the boundary and hence are determined by their values on the $\mathbf{x}_i$ (the dots).

$N(\mathbf{v})$ and $D(\mathbf{v})$ are indicated in Appendix I, where it is also shown that

$$D(\mathbf{v}) \geqq h^2(\mathbf{v}, \mathbf{v}) - h^2 N(\mathbf{v})/4 . \tag{2.1}$$

We now substitute the averaged eigenfunction $\mathbf{u} = \{\iint_{S_i} u\,dA/h^2\} \equiv \{u_i\}$, where $S_i = S(\mathbf{x}_i)$. *For the time being* we assume $u(\lambda)$ is the first normalized eigenfunction (eigenvalue) of the *full* L. To show

$$h^2(\mathbf{u}, \mathbf{u}) \geqq 1 - h^2\lambda/\pi^2 \tag{2.2}$$

we sum the inequality

$$h^2 u_i^2 \equiv h^2\left(\iint_{S_i} \frac{u\,dA}{h^2}\right)^2 \geqq \iint_{S_i} u^2 dA - \frac{h^2}{\pi^2}\iint_{S_i} |\nabla u|^2 dA \tag{2.21}$$

and note the union of the S_i is L.

This last inequality is a result of Weinberger [3, p. 342] and follows from the fact that $u - u_i$ is orthogonal over S_i to all constant functions. Hence, it may be substituted in the Rayleigh quotient for the first nontrivial Neumann eigenfunction for S_i, yielding

$$\frac{\pi^2}{h^2} \leqq \frac{\int\int_{S_i} |\nabla u|^2 dA}{(\int\int_{S_i} u^2 dA - h^2 u_i^2)},$$

which is equivalent to (2.21).

It remains to show $N(\mathbf{u}) \leqq \lambda$.

$N(\mathbf{u})$ may be regarded as composed primarily of sums of squares of horizontal and vertical differences of the $\bar{u}(\mathbf{x}_i)$. To estimate them, let H be any horizontal line through the interior mesh points, extending to the boundary (Fig. 3).

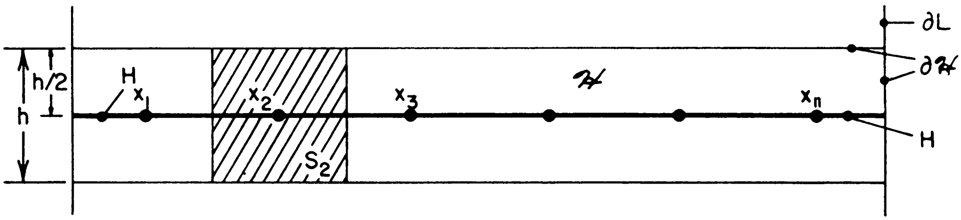


FIGURE 3

Regard the $\mathbf{x}_i$ in H as ordered $1 \leqq i \leqq n$ from left to right. Then set

$$\sigma_H = 2\bar{u}^2(\mathbf{x}_1) + \sum_{i=1}^{n-1} (\bar{u}(\mathbf{x}_{i+1}) - \bar{u}(\mathbf{x}_i))^2 + 2\bar{u}^2(\mathbf{x}_n).$$

Let V be a similar vertical line, and define σ_V similarly. Appendix I shows that $N(\mathbf{u}) \leqq \sum_{\text{all } H} \sigma_H + \sum_{\text{all } V} \sigma_V + \bar{u}^2(\mathbf{x}_*)$ where $\mathbf{x}_*$ is the mesh point NE of the re-entrant corner (see Fig. 2).

Furthermore, let $\mathcal{H}$ be the strip of height h centered on H. Then $\bigcup_{\text{all } \mathcal{H}} \mathcal{H} = L$ and, similarly, $\bigcup_{\text{all } \mathcal{V}} \mathcal{V} = L$.

We now show $\sigma_H \leqq \int\int_{\mathcal{H}} u_x^2 dA$. From this it follows that (since the $\mathcal{H}$ are disjoint) $\sum_{\text{all } H} \sigma_H \leqq \int\int_L u_x^2 dA$, and similarly $\sum_{\text{all } V} \sigma_V \leqq \int\int_L u_y^2 dA$. Hence

$$N(\mathbf{u}) \leqq \lambda + \bar{u}^2(\mathbf{x}_*). \tag{2.3}$$

Proof that $\sigma_H \leqq \int\int_{\mathcal{H}} u_x^2 dA$.

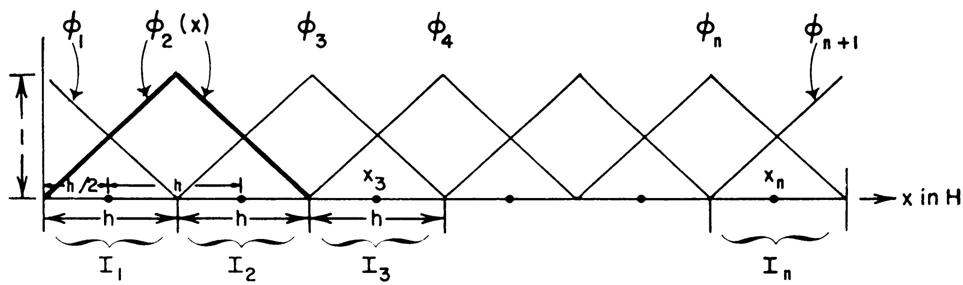


FIGURE 4

With Weinberger [3, p. 343], define $\phi_i(x)$, $i = 1, 2, \cdots, n+1$ as in Fig. 4. Observe that $|\phi_x| = 1/h$. Hence, integrating by parts,

$$\int_{I_{i+1}} \frac{u}{h}\, dx - \int_{I_i} \frac{u}{h}\, dx = -\int_{I_{i+1} \cup I_i} u \frac{d\phi_{i+1}}{dx} = \int_{I_{i+1} \cup I_i} u_x \phi_{i+1} dx \; ; \qquad i = 1, 2, \cdots, n\, .$$

Dividing by h, integrating on y with y going from 0 to h on $\mathcal{K}$, and squaring, we have

$$\left(\int_0^h \int_{I_{i+1}} \frac{u}{h^2}\, dxdy - \int_0^h \int_{I_i} \frac{u}{h^2}\, dxdy\right)^2 = \frac{1}{h^2}\left(\int_0^h \int_{I_{i+1} \cup I_i} \phi_{i+1} u_x dxdy\right)^2$$
$$\leqq \frac{1}{h^2} \int_0^h \int_{I_{i+1} \cup I_i} \phi_{i+1} dA \iint \phi_{i+1} u_x^2 dA\, ,$$

the last by Schwarz inequality. Since

$$\int_0^h \int_{I_{i+1} \cup I_i} \phi_{i+1} dA = h^2$$

we have

$$(\bar{u}(\mathbf{x}_{i+1}) - \bar{u}(\mathbf{x}_i))^2 \leqq \int_0^h \int_{I_{i+1} \cup I_i} \phi_{i+1} u_x^2 dA\, .$$

As for the ends (take the left one),

$$\int_{I_1} \frac{u}{h}\, dx = -\int_{I_1} u \frac{d\phi_1}{dx} = \int_{I_1} u_x \phi_1 dx\, ,$$

where we have used the fact that $u = 0$ at left boundary of $\mathcal{K}$; i.e., the left end of I_1. Hence

$$2\bar{u}^2(\mathbf{x}_1) = \frac{2}{h^2}\left(\int_0^h \int_{I_1} u_x \phi_1 dA\right)^2 \leqq \frac{2}{h^2} \iint \phi_1 dA \iint \phi_1 u_x^2 dA = \int_0^h \int_{I_1} \phi_1 u_x^2 dA\, .$$

Summing, we have, since $\sum \phi_i \equiv 1$, $\sigma_H \leqq \iint_{\mathcal{K}} u_x^2 dA$.

We now conclude from (2.1), (2.2), (2.3), with $\mathbf{x}_*$ as in Fig. 2,

$$\lambda_h \leqq \frac{\lambda + \bar{u}^2(\mathbf{x}_*)}{1 - c\lambda h^2 - \bar{u}^2(\mathbf{x}_*) h^2/4}\, .$$

Since $\bar{u}^2(\mathbf{x}_*)$ is $O(h^{4/3})$ for the first eigenfunction of the full L (with unknown constant), we cannot obtain an explicit lower bound for this region. However, this does give an $O(h^{4/3})$ upper bound on $\lambda_h - \lambda$, in analogy to that obtained in [11, Theorem 1, p. 1038], [4, p. 82] for the eigenvalue of the usual discrete Laplacian.

However, if we consider the first eigenfunction of the half-L, $\mathcal{L}$; and redefine u to be this eigenfunction reflected oddly in the diagonal of L, we have $\bar{u}(\mathbf{x}_*) = 0$. We assume u is normalized over the *full* L.

Redefining $\lambda_h = \min_{V_2} N(\mathbf{v})/D(\mathbf{v})$, where the $\mathbf{v} \in V_2$ are antisymmetric in the diagonal and vanish on it, we have, by (2.1), (2.2), and (2.3),

(2.4) $$\lambda \leqq \lambda_h \leqq \lambda/(1 - ch^2\lambda)\, , \qquad c = 1/4 + 1/\pi^2\, .$$

Thus, for h sufficiently small, $1 - ch^2\lambda_h > 0$, and we may conclude

$$\lambda_h/(1 + c\lambda_h h^2) \leqq \lambda \leqq \lambda_h , \qquad \lambda = \text{1st eigenvalue of } \mathcal{L} . \tag{2.5}$$

It is a straightforward extension of these arguments to conclude that the first eigenvalue of a horizontally stretched $\mathcal{L}$, approximated by the Rayleigh-Ritz technique using similarly stretched triangles of dimension $h \times k$, satisfies

$$\lambda_h/\{1 + c[\max (h, k)]^2\lambda_h\} \leqq \lambda \leqq \lambda_h .$$

3. $\Delta u = -\lambda u$, $u = 0$ **on** $\partial\mathcal{L}$; **Standard Mesh.** We place the standard mesh on $\mathcal{L}$ and triangulate as in Fig. 5:

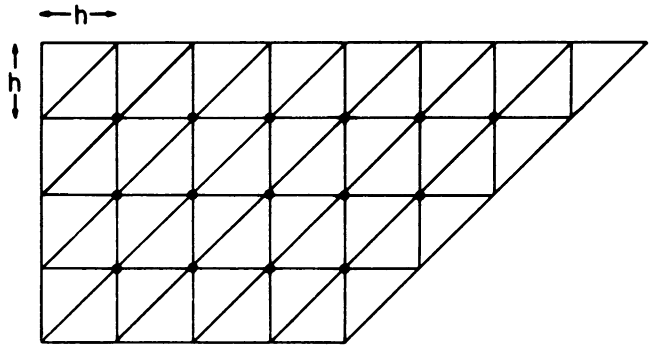


FIGURE 5

The quadratic forms $N(\mathbf{v})$ and $D(\mathbf{v})$ have changed (Appendix II), and the relations (1.1)–(1.3) must be re-proved. Appendix II shows again, however, that

$$D(\mathbf{v}) \geqq h^2(\mathbf{v}, \mathbf{v}) - h^2N(\mathbf{v})/4 . \tag{3.1}$$

We have difficulty in proving (1.2) because $R = \bigcup_{\mathbf{x}_i} S(\mathbf{x}_i) \neq \mathcal{L}$. R is shaded in Fig. 6.

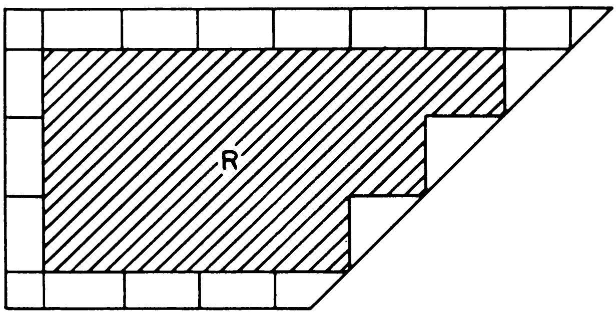


FIGURE 6

Let $\bar{R} = \mathcal{L} - R$. By (2.21) we have

$$h^2(\mathbf{u}, \mathbf{u}) \geqq \iint_R u^2 dA - \frac{h^2}{\pi^2}\iint_R |\nabla u|^2 dA$$
$$= 1 - \frac{h^2}{\pi^2}\lambda - \left(\iint_{\bar{R}} u^2 dA - \frac{h^2}{\pi^2}\iint_{\bar{R}} |\nabla u|^2 dA\right).$$

We now show the term in parentheses is negative. For we observe $\bar{R}$ is the union of four kinds of disjoint regions (Fig. 6): (1) $h/2 \times h$ rectangles, (2) $h/4 \times h/4$ squares, (3) $h \times h$ right triangles, (4) $h/2$-size $\mathcal{L}$-shaped regions. On each of these regions, R_i, u vanishes on an appropriate side, enabling one to reflect u negatively in this side and to extend its definition to an $h \times h$ square on which the extension has mean value zero. The argument leading to (2.21) now applies and yields, for each R_i,

$$\iint_{R_i} u^2 dA - \frac{h^2}{\pi^2} \iint_{R_i} |\nabla u|^2 dA \leqq 0 .$$

We conclude that the parenthetical term above is indeed negative, and hence

$$h^2(\mathbf{u}, \mathbf{u}) \geqq 1 - h^2\lambda/\pi^2 . \tag{3.2}$$

To prove (1.3) we examine the new numerator. σ_H may be redefined as

$$\bar{u}^2(\mathbf{x}_1) + \sum_{i=1}^{n-1} (\bar{u}(\mathbf{x}_{i+1}) - \bar{u}(\mathbf{x}_i))^2 + \bar{u}^2(\mathbf{x}_n)$$

and σ_V similarly. Appendix II shows that this time $N(\mathbf{u}) = \sum \sigma_H + \sum \sigma_V$.

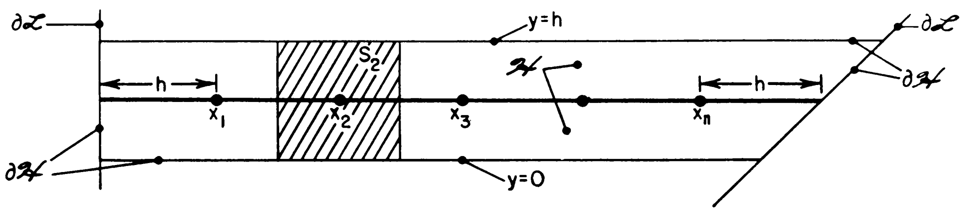


FIGURE 7

$\mathcal{H}$ is no longer the union of the $S(\mathbf{x}_i)$, $\mathbf{x}_i \in H$. We define new ϕ_i (Fig. 8):

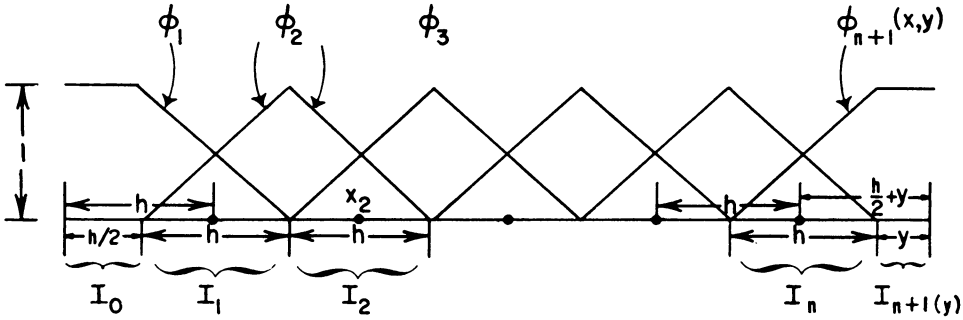


FIGURE 8

N.B.: length $(I_{n+1}) = y$, $y \in [0, h]$, and that ϕ_{n+1} is a function of x *and* y. We have to account mainly for $\bar{u}^2(\mathbf{x}_1)$ and for $\bar{u}^2(\mathbf{x}_n)$.

$$\int_{I_1} \frac{u}{h} dx = - \int_{I_0 \cup I_1} u \frac{d\phi_1}{dx} = \int_{I_0 \cup I_1} u_x \phi_1 dx ,$$

because $u \equiv 0$ on the left side of $\partial\mathcal{H}$.

Hence

$$\bar{u}^2(\mathbf{x}_1) = \int_0^h \int_{I_1} \frac{udA}{h^2} \leqq \frac{1}{h^2} \int_0^h \int_{I_0 \cup I_1} \phi_1 dA \int\int u_x^2 \phi_1 dA = \int_0^h \int_{I_0 \cup I_1} u_x^2 \phi_1 dA \ .$$

Similarly, because $u \equiv 0$ on the right side of $\partial\mathcal{H}$,

$$\int_{I_n} \frac{u}{h} dx = -\int_{I_n \cup I_{n+1}(y)} u \frac{\partial \phi_{n+1}(x, y)}{\partial x} dx = \int_{I_n \cup I_{n+1}(y)} u_x \phi_{n+1}(x, y) dx$$

and thus

$$\begin{aligned} \bar{u}^2(\mathbf{x}_n) &= \left(\int_0^h \int_{I_n} \frac{u}{h^2} dxdy \right)^2 \\ &\leqq \frac{1}{h^2} \int_0^h \int_{I_n \cup I_{n+1}(y)} \phi_{n+1}(x, y) dxdy \int_0^h \int_{I_n \cup I_{n+1}(y)} \phi_{n+1}(x, y) u_x^2 dxdy \\ &= \int_0^h \int_{I_n \cup I_{n+1}(y)} \phi_{n+1}(x, y) u_x^2 dxdy \ . \end{aligned}$$

The interior part of σ_H is handled as before; again the ϕ's add to 1 on $\mathcal{H}$ and

$$\sigma_H \leqq \iint_{\mathcal{H}} u_x^2 dxdy \ .$$

It should be clear how to define the ϕ's to verify, similarly, that

$$\sigma_V \leqq \iint_{\mathcal{V}} u_y^2 dxdy.$$

Summing, it follows again that

(3.3) $$N(\mathbf{u}) \leqq \lambda \ .$$

Hence, with λ the first eigenvalue of the $\mathcal{L}$,

(3.4) $$\lambda \leqq \lambda_h \leqq \lambda/(1 - ch^2\lambda) \ , \qquad c = 1/4 + 1/\pi^2 \ .$$

For the full L one runs into trouble with the strip $\mathcal{H}$ in Fig. 9, for $u \not\equiv 0$ on the left side of $\partial\mathcal{H}$. We have been unable to modify the procedure leading to (3.3) to conclude even that $N(\mathbf{u}) \leqq \lambda + O(h^2)$.

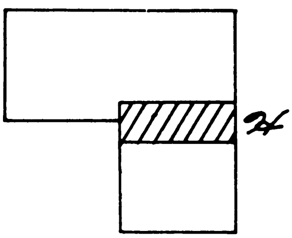


FIGURE 9

4. $\Delta u = -\lambda u$ in $\mathcal{L}$, $\partial u/\partial \nu = 0$ on $\partial\mathcal{L}$. We use a full L as indicated in Figure 10, where the level lines of the piecewise linear functions to be used have also been sketched. *All functions are assumed symmetric in the diagonal and orthogonal to the constant functions.*

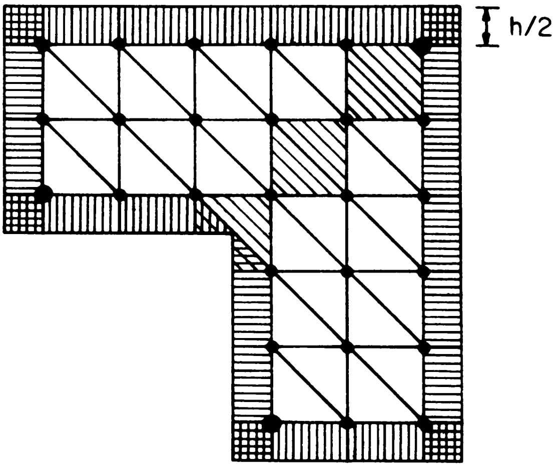


FIGURE 10

Appendix III shows two things: if $V_M = \max$ ($|v|$ at the three enlarged dots in Fig. 10), then

$$D(\mathbf{v}) \geqq h^2(\mathbf{v}, \mathbf{v}) - h^2[N(\mathbf{v}) + V_M{}^2]/4 \; ; \tag{4.1}$$

and that $N(\mathbf{v})$ consists *only* of a sum of squares of differences of neighboring values of $\mathbf{v}$. Thus, if u is normalized over the full L,

$$N(\mathbf{u}) \leqq \lambda \tag{4.3}$$

by the method of Section 2. (Since $N(\mathbf{u})$ contains only squared differences, no information about u on ∂L is required.) Also by Section 2,

$$h^2(\mathbf{u}, \mathbf{u}) \geqq 1 - h^2\lambda/\pi^2 \; . \tag{4.2}$$

Appendix IV shows that the mean values of u (over the $h \times h$ squares centered on the three dots) are bounded by an explicit increasing function of λ, $f(\lambda, h)$ (Eq. (IV.2)). From this and (4.1)–(4.3) we conclude

$$\lambda \leqq \lambda_h \leqq \lambda/[1 - ch^2\lambda - h^2 f(\lambda, h)/4] \, , \quad c = 1/4 + 1/\pi^2. \tag{4.4}$$

5. The Higher Eigenvalues. The min-max principle is, with $\sum$ running from 1 to k on i,

$$\lambda^{(k)} = \min_{w_1 \cdots w_k} \max_{a_1 \cdots a_k} \frac{\iint |\nabla \sum a_i w_i|^2 dA}{\iint (\sum a_i w_i)^2 dA} \, ,$$

where $w_1 \cdots w_k$ are piecewise differentiable, linearly independent functions over the domain which satisfy the boundary conditions. A larger min-max is obtained by restricting attention to all sets $\{v_1, \cdots, v_k\}$ of linearly independent piecewise linear functions satisfying the boundary conditions; and an upper bound $\lambda_h{}^{(k)}$ results:

$$\lambda^{(k)} \leqq \min_{v_1\cdots v_k} \max_{a_1\cdots a_k} \frac{\iint |\nabla \sum a_i v_i|^2 dA}{\iint (\sum a_i v_i)^2 dA} = \min_{\mathbf{v}_1\cdots \mathbf{v}_k} \max_{a_1\cdots a_k} \frac{N(\sum a_i \mathbf{v}_i)}{D(\sum a_i \mathbf{v}_i)} \equiv \lambda_h^{(k)}$$

(one notes for the last that $\mathbf{v}_1 \cdots \mathbf{v}_k$ are linearly independent if and only if $v_1 \cdots v_k$ are linearly independent). $\lambda_h^{(k)}$ is also the kth eigenvalue of the problem $A\mathbf{v} = \lambda B\mathbf{v}$; this can be seen by noting that $B^{-1}A$ is self-adjoint using $(\mathbf{v}, \mathbf{w})_B \equiv (\mathbf{v}, B\mathbf{w})$ and applying [20, p. 181]. Now set $\mathbf{u}_i = \{\iint_{S_j} u_i/h^2\}$ with u_i the ith eigenfunction normalized so that $\iint u_i^2 dA = 1$ and orthogonalized so $\iint u_m u_n dA = 0$, $m \neq n$.

Under the assumption that $\mathbf{u}_1 \cdots \mathbf{u}_k$ are linearly independent we have

$$\lambda_h^{(k)} \leqq \max_{a_1\cdots a_k;\, \Sigma a_i^2 = 1} \frac{N(\sum a_i \mathbf{u}_i)}{D(\sum a_i \mathbf{u}_i)}.$$

By the previous work in Sections 2 and 3, for each $a_1, \cdots, a_k$

$$\frac{N(\sum a_i \mathbf{u}_i)}{D(\sum a_i \mathbf{u}_i)} \leqq \frac{\iint |\nabla \sum a_i u_i|^2 dA}{\iint (\sum a_i u_i)^2 dA - ch^2 \iint |\nabla \sum a_i u_i|^2 dA} = \frac{\sum \lambda^{(i)} a_i^2}{1 - ch^2 \sum \lambda^{(i)} a_i^2}.$$

Hence

$$\lambda_h^{(k)} \leqq \max_{a_1\cdots a_k;\, \Sigma a_i^2 = 1} \frac{\sum \lambda^{(i)} a_i^2}{1 - ch^2 \sum \lambda^{(i)} a_i^2}.$$

But $0 \leqq f \leqq f_{\max}$ and $k > 0$ implies $\max [f/(1 - kf)] = f_{\max}/(1 - kf_{\max})$. Hence

$$\lambda^{(k)} \leqq \lambda_h^{(k)} \leqq \lambda^{(k)}/(1 - ch^2 \lambda^{(k)})$$

and (1.6) follows.

Finally, we show that for fixed k, $\mathbf{u}_1 \cdots \mathbf{u}_k$ are linearly independent as $h \to 0$. For suppose $\sum a_i \mathbf{u}_i = \mathbf{0}$, $\sum a_i^2 = 1$. From (1.1) and the procedures leading to (1.2)

$$\begin{aligned} D(\textstyle\sum a_i \mathbf{u}_i) &\geqq \iint (\textstyle\sum a_i u_i)^2 dA - ch^2 \iint |\nabla \textstyle\sum a_i u_i|^2 dA \\ &\geqq 1 - ch^2 \textstyle\sum \lambda^{(i)} a_i^2, \quad c = 1/4 + 1/\pi^2. \end{aligned}$$

Let $ch^2 < 1/\lambda^{(k)}$. Then $D(\sum a_i \mathbf{u}_i) > 0$. But D is positive definite, hence $\sum a_i \mathbf{u}_i \neq \mathbf{0}$, a contradiction of the linear dependence of the $\mathbf{u}_i$, $i = 1, \cdots, k$.

It is easily verified that the additional term, introduced in the denominators when considering the "Neumann" eigenfunctions of Section 4, causes little difficulty, and results in

$$\lambda^{(k)} \leqq \lambda_h^{(k)} \leqq \lambda^{(k)} \Big/ \left[1 - ch^2 - h^2 \sum_{i=1}^{k} f(\lambda^{(i)}, h)/4\right] \equiv g_h^{(k)}(\lambda^{(1)}, \cdots, \lambda^{(k)}) \tag{5.1}$$

6. Upper Bound Using Standard Matrix Eigenvalues. The usual approach to the membrane's eigenvalues is to solve the matrix eigenvalue problem

$$A\mathbf{v}^* = \lambda_h^* h^2 \mathbf{v}^* \tag{6.1}$$

instead of the problem indicated in Section 1 as $A\mathbf{v} = \lambda_h B\mathbf{v}$.

Since $\mathbf{v}^*$ may be substituted in the Rayleigh quotient, we have by (1.1), that

$$\begin{aligned} \lambda &\leqq N(\mathbf{v}^*)/D(\mathbf{v}^*) \leqq N(\mathbf{v}^*)/[h^2(\mathbf{v}^*, \mathbf{v}^*) - h^2 N(\mathbf{v}^*)/4] \\ &= \lambda_h^*/(1 - h^2 \lambda_h^*/4), \end{aligned} \tag{6.2}$$

giving a strict upper bound on the lowest eigenvalue. For the mesh of Section 3, [4, p. 68] shows

$$\lambda_h^* = \lambda - \gamma h^2 + o(h^2)\,, \qquad h \to 0\,;$$

thus λ_h^* is an asymptotic lower bound, not a strict lower bound.

7. A Numerical Example. The lowest eigenvalues of both the $\mathcal{L}$ and the L, corresponding to $u = 0$ on the boundary, were approximated by the Rayleigh-Ritz technique using the mesh of Section 3. A suitable generalization of the successive overrelaxation technique used by Moler [4, pp. 114–117] was programmed to solve $A\mathbf{v} = \lambda_h B\mathbf{v}$. The Rayleigh quotients were computed in double precision to obtain sufficient accuracy in the matrix eigenvalues. Values of $1/h$ were 8(4)36 and 42 (Table 1).

TABLE 1

Rayleigh-Ritz eigenvalues, standard mesh, $u = 0$ on boundary

Half-L, $\mathcal{L}$				L
$1/h$	Upper bound λ_h	Lower bound $\lambda_h/(1 + ch^2\lambda_h)$	Difference	Upper bound λ_h
8	15.5572882	14.334	1.224	9.9659766
12	15.3580389	14.804	0.555	9.8028565
16	15.2879549	14.974	0.314	9.4081708
20	15.2554144	15.054	0.202	9.7099600
24	15.2376994	15.098	0.140	9.6921083
28	15.2270005	15.124	0.103	9.6807234
32	15.2200476	15.141	0.079	9.6729507
36	15.2152759	15.153	0.062	9.6673700
42	15.2105050	15.165	0.046	9.6615029

TABLE 2

Previously computed standard eigenvalues, source indicated

Half-L, $\mathcal{L}$ (Bowdler & Wilkinson)			L (mostly Moler)	
$1/h$	λ_h^*	Upper bound $\lambda_h^*/(1 - h^2\lambda_h^*/4)$	$1/h$	λ_h^*
6	14.8325923	16.536	4	9.64142546 (B&W)
8	14.9931528	15.926	8	9.69316221 (B&W)
10	15.0671631	15.657	10	9.68829145
11	15.0899260	15.576	20	9.66696983
12	15.1072074	15.515	30	9.65743368
13	15.1206339	15.467	40	9.65249358
14	15.1312711	15.430	50	9.64954711
15	15.1398404	15.399	80	9.64527693
16	15.1468447	15.375	100	9.64393241

The lowest eigenvalue of (6.1) has been computed for the L by Moler [4, p. 124] with $1/h = 10(10)100$. Bowdler and Wilkinson at the National Physical Laboratory calculated the lowest eigenvalue of (6.1) for the $\mathcal{L}$ and the L (unpublished, 1962) with $1/h = 6(1)16$. See also [12, p. 841 ff], where two methods are discussed which significantly improve on the order of convergence to the first eigenvalue of the L (Table 2).

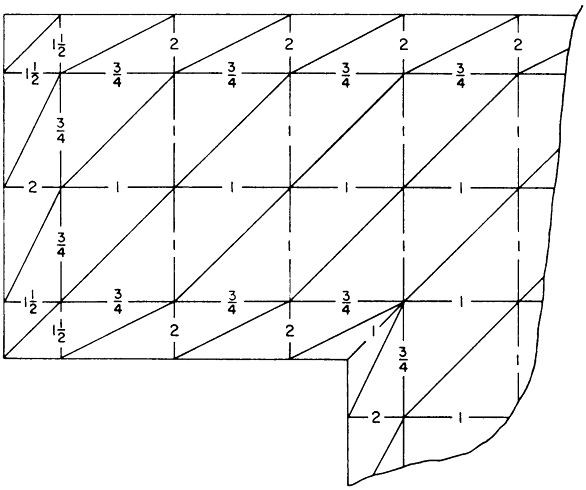
To more easily compare these results, all four sets of eigenvalues were fit, as functions of h, by a nonlinear least-squares routine which could handle approximations of the form $\sum a_i h^{b_i}$, a_i and b_i parameters. The best fits were

Source	*Region*	*Fit to* $\lambda_h = \lambda(h)$
Moler	L	$\lambda_L + 2.2h^{4/3} - 5.2h^2 - h^{3.1}$
Author	L	$\lambda_L + 2.2h^{4/3} + 11.8h^2 + 11h^{3.6}$
NPL	$\mathcal{L}$	$\lambda_{\mathcal{L}} - 13h^2 + h^3$
Author	$\mathcal{L}$	$\lambda_{\mathcal{L}} + 23h^2 - 2h^3$

Assuming $\lambda_{\mathcal{L}}$ known (below), (3.4) predicts *a priori* an upper bound behaving like $\lambda_{\mathcal{L}} + 81h^2$. $\lambda_{\mathcal{L}}$ was 15.197 for the NPL eigenvalues and 15.1973 for the author's, while λ_L was 9.639724 for both Moler's and the author's eigenvalues. From the results for the $\mathcal{L}$ it is easily seen that the *a posteriori* lower bound derived from (3.4) behaves like $\lambda - 58h^2$ while the *a posteriori* upper bound (6.2) behaves like $\lambda + 44h^2$.

Thanks are due to Connie Luders for preparing the diagrams and to the referee for his help, especially with Appendix IV.

APPENDIX I. *Numerator as a sum of squares of differences*:
Notation: +—— 2 ——+ means $2[v(\text{right}) - v(\text{left})]^2$ occurs in the numerator.



This is easily verified by squaring the (constant) gradient above each triangle, multiplying by the area of the triangle, and adding up over the triangles.

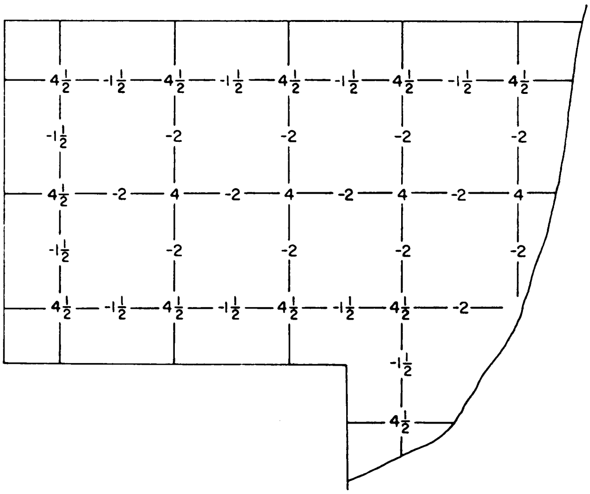
There is no loss in generality in considering the case of the large h shown here and in what follows, for the forms do not change with decreasing h.

All diagrams which follow (in the next two appendices as well as this one) consider various quadratic forms using the notation to be described next.

Numerator as a quadratic form:

Notation: —— 4 —— means 4 v^2(vertex) occurs in the form;

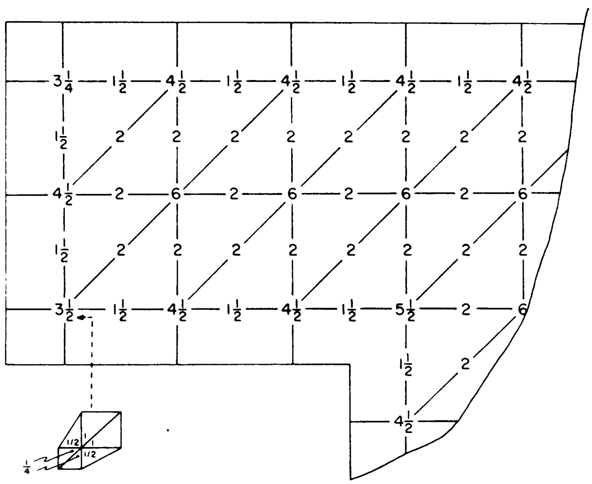
—— —— -2 —— —— means $-2v$(left)v(right) occurs in the form.



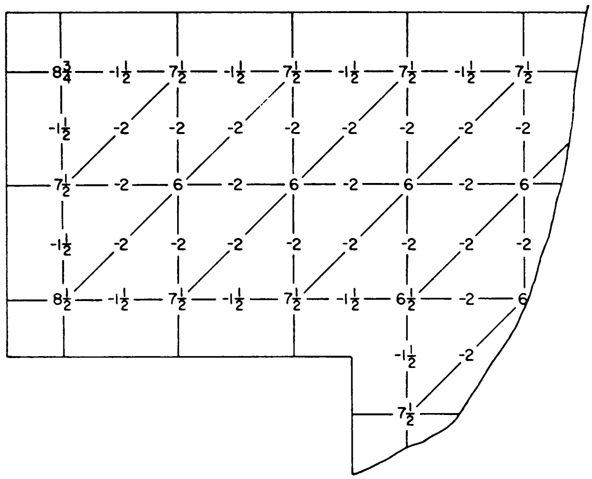
The denominator. If a linear function, $P(\mathbf{x})$, defined on a triangle, T, of base h and altitude k, has values a, b, and c above each vertex then

$$\iint_T P^2 dA = \frac{hk}{12}(a^2 + b^2 + c^2 + ab + ac + bc) .$$

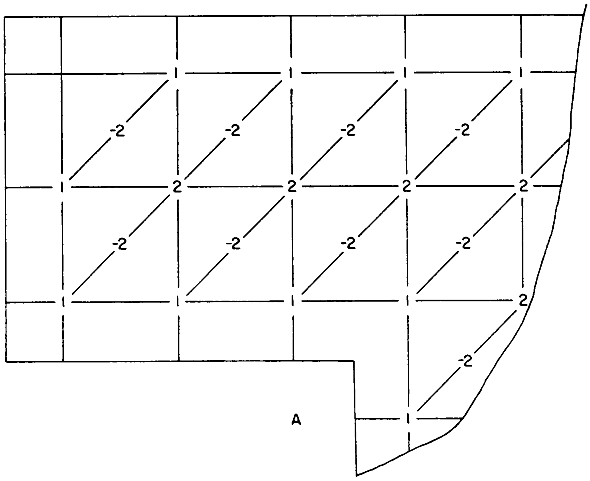
For example: to find the coefficient of v^2 ("SW corner") we examine the contribution of the six triangles containing that corner (see insert). Hence the denominator is h^2 (following form)/12:

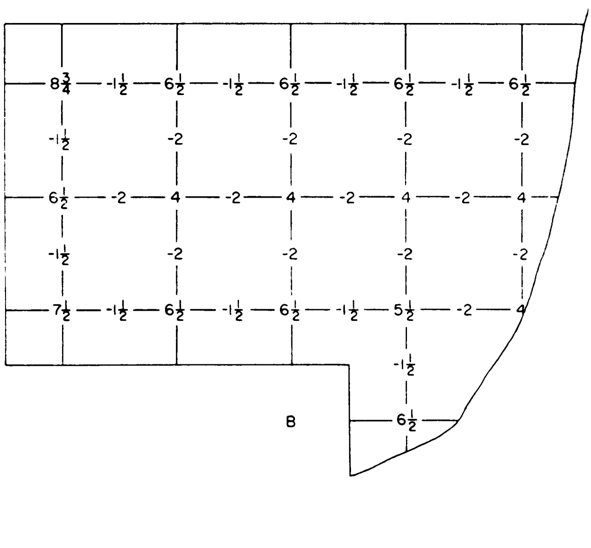


Thus $h^2(\mathbf{v}, \mathbf{v}) = D(\mathbf{v}) + h^2$ (following form)/12:

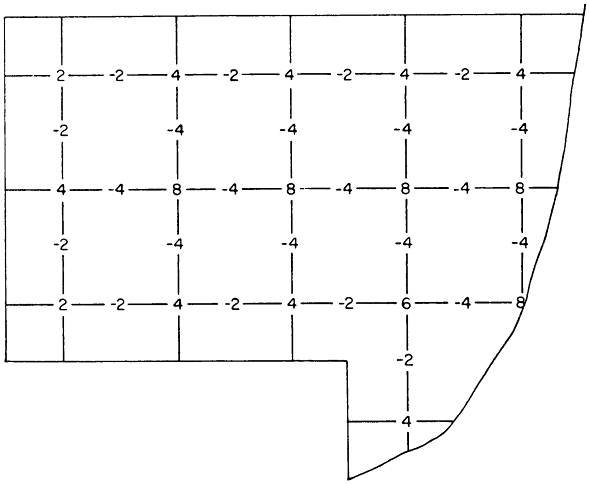


This last form is the sum of $A + B$, where $A = \sum$ (diagonal differences)^2.

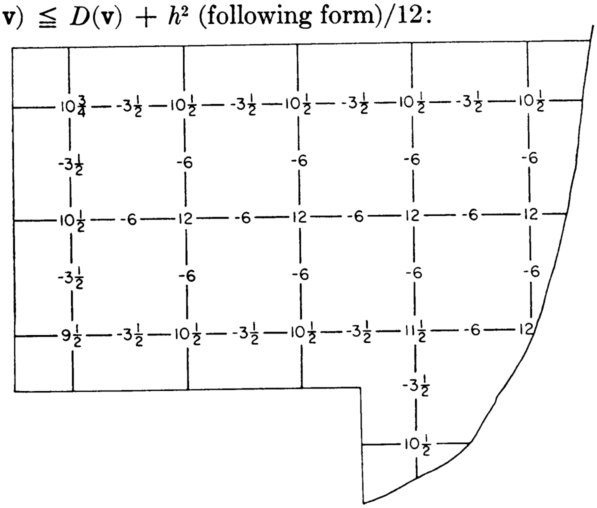


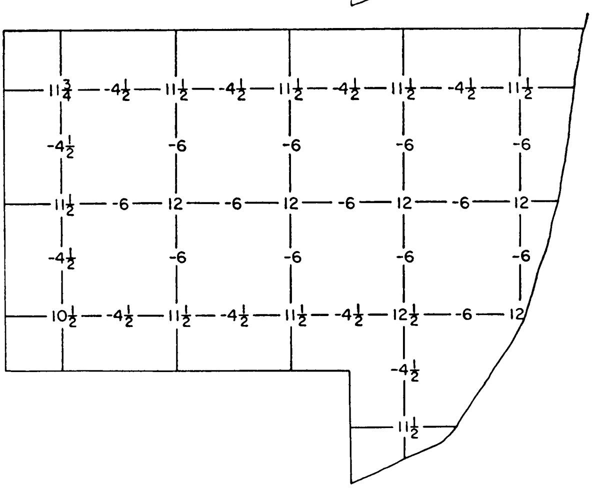


From the inequality $(a - c)^2 \leqq 2[(a - b)^2 + (b - c)^2]$, or $(a - c)^2 \leqq (a - d)^2 + (d - b)^2 + (a - b)^2 + (b - c)^2$, $A \leqq$ following form:



Hence $h^2(\mathbf{v}, \mathbf{v}) \leqq D(\mathbf{v}) + h^2$ (following form)/12:



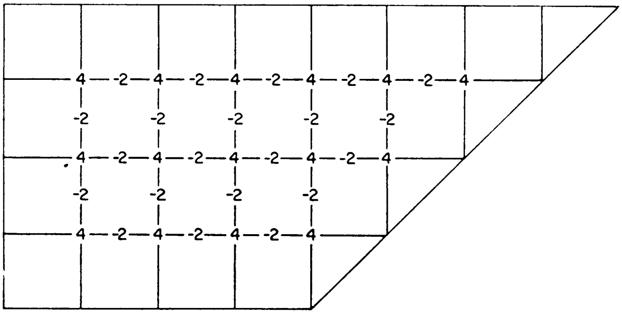


Now, adding $\frac{1}{2}\sum_{\mathrm{bdry}}$ ("boundary differences")^2, where a "boundary difference" is the difference of values at two neighboring points, each adjacent to the boundary, $h^2(\mathbf{v}, \mathbf{v}) \leqq D(\mathbf{v}) + h^2$ (preceding form)/12.

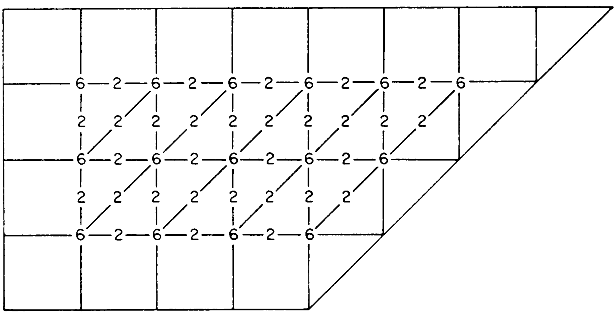
Finally, by adding appropriate squares of "boundary values," we have

$$h^2(\mathbf{v}, \mathbf{v}) \leqq D(\mathbf{v}) + h^2(3\cdot\text{Numerator})/12\,, \quad \text{or}$$
$$D(\mathbf{v}) \geqq h^2(\mathbf{v}, \mathbf{v}) - h^2N(\mathbf{v})/4\,.$$

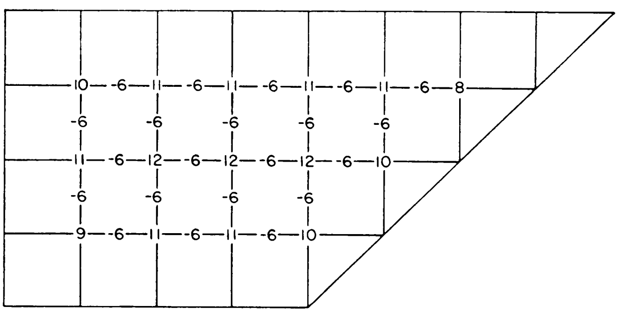
APPENDIX II. *Numerator as a quadratic form.* With the notation of Appendix I, $N(\mathbf{v})$ is the following form:



Denominator as a quadratic form. $D(\mathbf{v}) = h^2$ (following form)/12:



Performing manipulations as in Appendix I we arrive at the conclusion that $h^2(\mathbf{v}, \mathbf{v}) \leqq D(\mathbf{v}) + h^2$ (following form)/12:



Hence

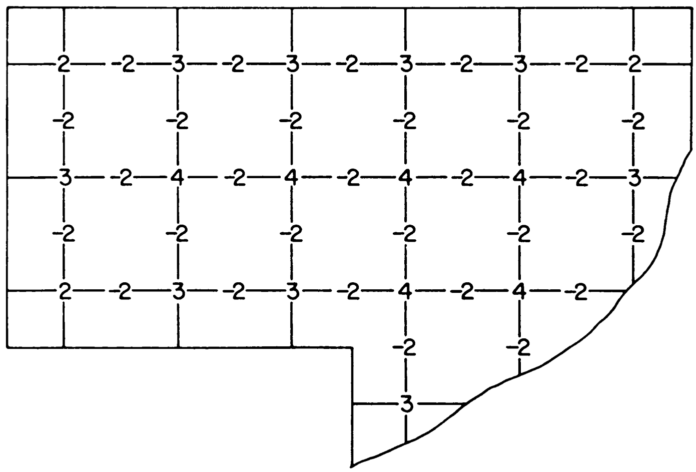
$$h^2(\mathbf{v}, \mathbf{v}) \leqq D(\mathbf{v}) + h^2(3\, N(\mathbf{v}))/12\,,$$

or

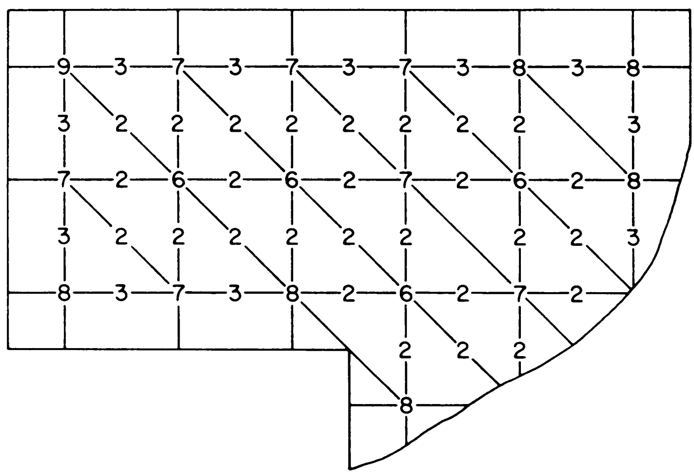
$$D(\mathbf{v}) \geqq h^2(\mathbf{v}, \mathbf{v}) - h^2N(\mathbf{v})/4\,.$$

APPENDIX III. In deriving and manipulating the quadratic forms to follow, careful advantage is taken of the level lines of $v(x)$ (Fig. 10) and the symmetry of $v(x)$ in the diagonal of L.

$N(\mathbf{v})$ is the following form:

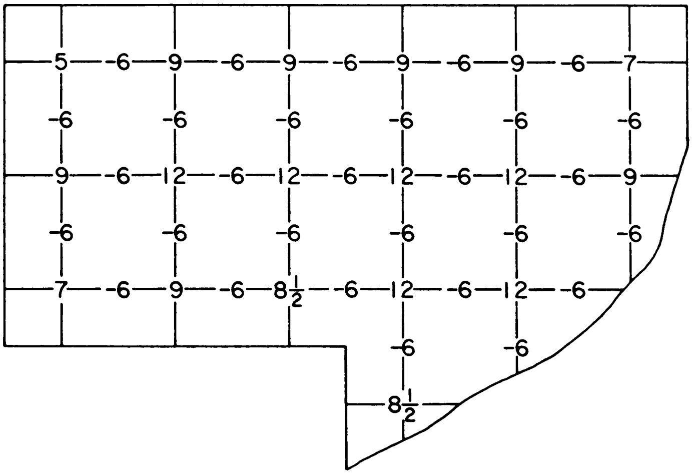


$D(\mathbf{v})$ is the following form:



Using some care in manipulating D as in Appendix I, one finds $h^2(\mathbf{v}, \mathbf{v}) \leqq D(\mathbf{v}) +$

h^2 (following form)/12:



Thus $h^2(\mathbf{v}, \mathbf{v}) \leqq D(\mathbf{v}) + h^2N(\mathbf{v})/4 + h^2V/12$, where V is the sum of the squares of $\mathbf{v}$ at the three heavy dots in Fig. 10. Equation (4.1) follows.

APPENDIX IV. *Bounds on Eigenfunctions.* It is easy to find explicit $O(\lambda)$ bounds $(\lambda \rightarrow \infty)$ on $\max_D |u|$ if u satisfies $\Delta u = -\lambda u$ in D, $u = 0$ on ∂D, $\int\int_D u^2 = 1$. For u satisfies an integral equation involving the Green's function of D, from which the Schwarz inequality yields $\max |u|^2 \leqq \lambda^2 \max \int\int G^2$. Because G is an increasing function of D, an explicit bound can be found by surrounding D with a rectangle and expanding the rectangle's Green's function in terms of its eigenfunctions. This $O(\lambda)$ bound can be improved to $\max |u| \leqq (4\lambda/\pi)^{1/2}$ by using the Green's function for $\Delta u - pu$, $p > 0$ and optimizing the resulting bound by choosing $p = \lambda$.

That such a bound for a general domain must be $O(\lambda^q)$ $(\lambda \rightarrow \infty)$ for some $q \geqq 1/4$ may be seen by considering the subsequence of normalized eigenfunctions for the unit circle: $J_0(\sqrt{\lambda_k}r)/[\sqrt{\pi}J_0'(\sqrt{\lambda_k})]$, $\sqrt{\lambda_k} = k$th zero of J_0, [15, p. 306], [18, p. 364].

When the boundary condition is $\partial u/\partial \nu = 0$ (instead of $u = 0$) the above method fails, for the behaviour of the Neumann's function (or even the kernel function) is not easily tied to the relation of the domains involved in their definition. Another procedure is called for, then, and is based on a mean-value theorem.

The mean-value theorem, for regular function, u, satisfying $\Delta u = -\lambda u$ inside a circle of radius r, is [16, p. 289, correcting a misprint]:

$$\int_0^{2\pi} u(r, \theta) r d\theta = 2\pi u(0, 0) r J_0(\sqrt{\lambda} r) .$$

Multiplying by $J_0(\sqrt{\lambda}r)$, integrating from 0 to R, and applying the Schwarz inequality, we have [15, p. 484, 11. 3.34]

$$\text{(IV.1)} \quad \begin{aligned} u^2(0, 0) &\leqq \int_0^R \int_0^{2\pi} u^2 r d\theta dr \Big/ \left[2\pi \int_0^R J_0^{\,2}(\sqrt{\lambda} r) r dr\right] \\ &= \int_0^R \int_0^{2\pi} u^2 r d\theta dr \big/ \{\pi R^2 [J_0^{\,2}(\sqrt{\lambda} R) + J_1^2\,(\sqrt{\lambda} R)]\} . \end{aligned}$$

(This bound is sharp for the eigenfunctions of the unit circle mentioned above.) It is of no use, however, to let $R \to 0$ to obtain a uniform bound on $|u|$; using the analogue (for circles) of (2.21) to bound the integral in (IV.1) (in terms of $u^2(0, 0)$ and λ) only yields $u^2(0, 0) \leqq u^2(0, 0) +$ something.

To obtain a usable bound, then, of $|u|$ at the three large dots in Fig. 10, let us first focus attention on the dot adjacent to the 90° corner. Reflection of the $\mathcal{L}$ (and u) three times around the corner defines a new region, $\mathcal{L}_4$, and a new function, $\tilde{u}$. Assuming $\Delta\tilde{u} = -\lambda\tilde{u}$ in $\mathcal{L}_4$ (this assumption is discussed below) one gains the analyticity of $\tilde{u}$ in $\mathcal{L}_4$ and assures the validity of (IV.1) for any disc contained in $\mathcal{L}_4$. Placing (0, 0) at the dot, and recalling u is normalized over the *full* L, (IV.1) yields $u^2(dot) \leqq 2/\{\pi R^2[J_0^2(\sqrt{\lambda}R) + J_1^2(\sqrt{\lambda}R)]\}$ for any $R < 1 - h$. Under similar assumptions, u at the other two dots may be bounded by reflecting three or seven times, the conclusion being

$$\text{(IV.2)} \qquad \begin{aligned} \max\,[u^2(\text{three dots})] &\leqq 4/\{\pi R^2[J_0^2(\sqrt{\lambda}R) + J_1^2(\sqrt{\lambda}R)]\} \\ &\equiv f(\lambda, h)\,, \qquad R = 1 - h\,. \end{aligned}$$

We see $f(0, h) = 4/[\pi(1 - h)^2]$; and that for fixed h, $f = O(\sqrt{\lambda})$, $\lambda \to \infty$. Furthermore, since $\partial f/\partial\sqrt{\lambda} = 2\pi R^2 J_1^2(\sqrt{\lambda}R)/(\sqrt{\lambda}f^2) \geqq 0$ (using $J_0'(z) = -J_1(z)$, $J_1'(z) = J_0(z) - J_1(z)/z$), we see f is increasing in λ (for fixed h) and so may be used in obtaining the lower bound (1.5a).

Finally, (IV.2) is seen to bound u uniformly in the $h \times h$ squares centered on the three dots, and thus bounds its average over these squares.

To show that $\Delta\tilde{u} = -\lambda\tilde{u}$ in $\mathcal{L}_4$, choose a disc D, centered on the 90° corner of $\mathcal{L}$, of radius small enough so that the lowest eigenvalue μ_1 of the problem $\Delta v = -\mu_1 v$ in D, $v = 0$ on ∂D satisfies $\mu_1 > \lambda$. Then the solution u^* to $\Delta u^* = -\lambda u^*$ in D, $u^* = \tilde{u}$ on ∂D, is unique and satisfies $\partial u^*/\partial\nu = 0$ on $\partial\mathcal{L} \cap D$. Since μ_2, the lowest eigenvalue of the problem $\Delta v = -\mu_2 v$ in $\mathcal{L} \cap D$, $v = 0$ on $\partial D \cap \mathcal{L}$, $\partial v/\partial\nu = 0$ on $\partial\mathcal{L} \cap D$ satisfies $\mu_2 > \mu_1 > \lambda$; $u^* \equiv \tilde{u}$ in $\mathcal{L} \cap D$, and thus in $\mathcal{L}_4 \cap D$. Similar arguments show $\tilde{u}$ is regular at every interior point of $\mathcal{L}_4$; and that the corresponding reflection around the 45° corner is also regular in its domain of eight $\mathcal{L}$'s.

Finally, it is worth pointing out that, had the hypotenuses of the triangles in Fig. 10 sloped like those in Fig. 2, it would have been necessary to bound $|u|$ in a neighborhood of the 135° corner; and the reflections would have had to have been completed on a Riemann surface (where no mean value theorem is known to the author).

For discussions of pointwise bounds in general see, e.g., [17, p. 101] and its references.

University of California
Los Alamos Scientific Laboratory
Los Alamos, New Mexico

1. R. Courant, "Variational methods for the solutions of problems of equilibrium and vibrations," *Bull. Amer. Math. Soc.*, v. 49, 1943, pp. 1–23. MR **4**, 200.

2. L. Collatz, "Konvergenz des Differenzverfahrens bei Eigenwertproblemen partieller Differentialgleichungen," *Deutsche Math.*, v. 3, 1938, pp. 200–212.

3. H. F. Weinberger, "Lower bounds for higher eigenvalues by finite difference methods," *Pacific J. Math.*, v. 8, 1958, pp. 339–368. MR **21** #6097.

4. C. Moler, *Finite Difference Methods for the Eigenvalues of Laplace's Operator*, Stanford Computer Science Report CS22, 1965. (Available from Defense Document Center, Cameron Station, Alexandria, Va.)

5. J. SYNGE, *The Hypercircle in Mathematical Physics: A Method for the Approximate Solution of Boundary Value Problems*, Cambridge Univ. Press, New York, 1957, Sections 3.5–3.8 ff. MR **20** #4073.

6. G. PÓLYA, "Sur une interprétation de la méthode des différences finies qui peut fournir des bornes supérieures ou inférieures," *C. R. Acad. Sci. Paris*, v. 235, 1952, pp. 995–997. MR **14,** 656.

7. G. PÓLYA, "Estimates of eigenvalues," *Studies in Mathematics and Mechanics Presented to Richard von Mises*, Academic Press, New York, 1954, pp. 200–207. MR **16,** 482.

8. G. BIRKHOFF, C. DE BOOR, B. SWARTZ, & B. WENDROFF, "Rayleigh-Ritz approximation by piecewise cubic polynomials," *SIAM J. Numer. Anal.*, v. 3, 1966, pp. 188–203. MR **34** #3773.

9. G. FORSYTHE & W. R. WASOW, *Finite-Difference Methods for Partial Differential Equations*, Wiley, New York, 1960, pp. 340–351. MR **23** #B3156.

10. B. HUBBARD, "Bounds for eigenvalues of the free and fixed membrane by finite difference methods," *Pacific J. Math.*, v. 11, 1961, pp. 559–590. MR **25** #4633.

11. L. VEĬDINGER, "Computation of the eigenvalues of a membrane by a finite-difference method," *Z. Vyčisl. Mat. i Mat. Fiz.*, v. 4, 1964, pp. 1037–1044 = *U.S.S.R. Comput. Math. and Math. Phys.*, v. 4, 1964. MR **30** #1620.

12. J. K. REID & J. E. WALSH, "An elliptic eigenvalue problem for a reentrant region," *J. Soc. Indust. Appl. Math.*, v. 13, 1965, pp. 837–850. MR **31** #6392.

13. J. HERSCH, "Equations différentielles et fonctions de cellules," *C. R. Acad. Sci. Paris*, v. 240, 1955, pp. 1602–1604. MR **16,** 929.

14. J. HERSCH, "Lower bounds for all eigenvalues by cell functions: a refined form of H. F. Weinberger's method," *Arch. Rational Mech. Anal.*, v. 12, 1963, pp. 361–366. MR **29** #745.

15. R. COURANT & D. HILBERT, *Methods of Mathematical Physics*, Vol. I, Interscience, New York, 1953. MR **16,** 426.

16. R. COURANT & D. HILBERT, *Methods of Mathematical Physics*, Vol. II, Interscience, New York, 1962. MR **25** #4216.

17. L. E. PAYNE & H. F. WEINBERGER, "Bounds for solutions of second order elliptic equations in terms of arbitrary vector fields," *Arch. Rational Mech. Anal.*, v. 20, 1965, pp. 95–106. MR **33** #401.

18. M. ABRAMOWITZ & I. A. STEGUN, (EDITORS), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards, Applied Math. Series, No. 55, U. S. Government Printing Office, Washington, D. C., reprint, 1965. MR **31** #1400.

19. H. WEINBERGER, "Upper and lower bounds for eigenvalues by finite difference methods," *Comm. Pure Appl. Math.*, v. 9, 1956, pp. 613–623. MR **18,** 826.

20. P. HALMOS, *Finite-Dimensional Vector Spaces*, Van Nostrand, Princeton, N. J., 1958. MR **19,** 725.