TECHNICAL NOTES AND SHORT PAPERS

Proof that Every Integer \( \leq 452,479,659 \) is a Sum of Five Numbers of the Form \( Q_x = \left( x^3 + 5x \right) / 6, x \geq 0 \)

By Herbert E. Salzer and Norman Levine

Watson [1] proved that every positive integer is a sum of eight tetrahedral numbers \( T_x = (x^3 - x) / 6, x \geq 1 \), as well as of eight numbers \( Q_x = T_x + x = (x^3 + 5x) / 6, x \geq 0 \), and states that "a similar result holds" for \( R_x = T_x - x = (x^3 - 7x) / 6, x = 0 \) or \( x \geq 3 \). He also points out that \( T_x, Q_x \) and \( R_x \) are the only expressions of the form \( T_x + Dx, D \) integral, which can take the value 1 and permit a universal result for summands \( \geq 0 \). In view of the results obtained by the authors in [2], which gave overwhelming evidence that every integer required only five values of \( T_x \), it is interesting to see whether a similar conjecture is justified for \( Q_x \) and \( R_x \).

There is an immediate lack of comparative interest in \( R_x \) whose nonnegative values are 0, 1, 6, 15, 29, 49, 76, 111, . . . because six such addends are needed for the following values of \( n \leq 100 \): 11, 26, 40, 54, 69. The remaining form of possible interest, namely \( Q_x \), whose values run 0, 1, 3, 7, 14, 25, 41, 63, 92, 129, 175, . . . does not appear offhand as promising or "nice looking" as \( T_x \) to allow every integer to be a sum of five, even though Watson [1] verified that for \( n \leq 210 \). However, it was quite a surprise to find that, defining an "exceptional number" as a number requiring more than four summands, when the test was made up to 1,000,000, for \( Q_x \) there were vastly fewer exceptional numbers than for \( T_x \). Thus, whereas in [1] the authors found as many as 241 exceptional numbers for \( T_x \), the largest being as high as 343,867, in the present investigation only 21 exceptional numbers were found for \( Q_x \), the largest being only 28415.

Following are the only numbers \( \leq 1,000,000 \) that are not the sum of four numbers \( Q_x \):

<table>
<thead>
<tr>
<th>Table I</th>
<th>Exceptional numbers ( \leq 1,000,000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>372</td>
</tr>
<tr>
<td>115</td>
<td>541</td>
</tr>
<tr>
<td>122</td>
<td>1805</td>
</tr>
<tr>
<td>166</td>
<td>2532</td>
</tr>
<tr>
<td>334</td>
<td>2773</td>
</tr>
</tbody>
</table>

From Table I it is immediately apparent that every integer \( \leq 1,000,000 \) is a sum of five numbers \( Q_x \). The size of the gap between 28415 and 1,000,000 enables us to find a number \( N \) much larger than 1,000,000 for which every \( n \leq N \) is a \( \sum_5 \), or sum of five numbers \( Q_x \). The basic principle in finding such an \( N \) is not new, having been employed by both Watson [1] and the authors [2] in a sort of loose manner. Apparently the sharpest form of that principle is formulated in the lemma below, which is also applicable to \( T_x \) and a wide class of similar functions.

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Lemma. Let $E$ be the largest exceptional number found in a test extending through $L > E$. Let $x$ be the largest $x$ for which $\Delta Q_x = Q_{x+1} - Q_x < I = L - E$. Suppose that from the tabulation of exceptional numbers it is apparent that every $n \leq E$ is a $\sum_5$. Then any $n \leq N = Q_{x+1} + L$ is a $\sum_5$.

Proof. For $n \leq L$, the result is in the hypothesis. If $L < n < Q_{x+1}$, $n - some Q_i, i \leq x - 1$, will come closest above $L$, so that $n - Q_{i+1} \leq L$. Since $Q_{i+1} - Q_i \leq Q_x - Q_{x-1} < Q_{x+1} - Q_x < I$, $n - Q_{i+1}$ falls within the interval $(E, L)$, so that $n$ is a $\sum_5$. For $n = Q_{x+1}$, or $N = Q_{x+1} + L$, the result is immediate, since $L$ is the largest tested $\sum_4$. For $Q_{x+1} < n < N = Q_{x+1} + L$, since $n - Q_{x+1} < L$, if $n > L$, $n - some Q_{i}, i \leq x$, comes closest above $L$, so that $n - Q_{i+1} \leq L$, and from $Q_{i+1} - Q_i \leq Q_{i+1} - Q_x < I$, $n - Q_{i+1}$ falls within the interval $(E, L)$, so that $n$ is a $\sum_5$. Q.E.D.

If we try to push the lemma to apply beyond $N = Q_{x+1} + L$, say up to $Q_{x+1} + L + e$, it fails because for some $n$ beyond $Q_{x+1} + L$ the $i$ making $n - Q_i$ come closest above $L$ must be $\geq x + 1$, and we have no assurance that $n - Q_{i+1}$ falls within the interval $(E, L)$. The reason is that $Q_{i+1} - Q_i \geq Q_{i+2} - Q_{x+1} \geq I$, and if the number by which $Q_{z+2} - Q_{z+1}$ exceeds $I$ is greater than the number by which $n - Q_i$ exceeds $L$, then $n - Q_{i+1} < L - I = E$.

Applying this lemma to $Q_2$, where the condition $\Delta Q_x < I$ is equivalent to $x^2 + x < 2I$, from Table I, $E = 28415$, $L = 1,000,000$, $2I = 2(L - E) = 1,943,170$, and $x = 1393$ is the largest $x$ for which $x^2 + x + 2 = 1,941,844 < 2I$. Thus, every $n \leq N = Q_{1194} + L = 451,479,659 + 1,000,000 = 452,479,659$ is a $\sum_5$.

We may apply this lemma also to $T_x$ for which it was found in [1] that $E = 343,867$ when the test for exceptional numbers extended as far as $L = 1,043,999$. From the tabulation of exceptional numbers in [1] it was apparent that every $n \leq E$ is a $\sum_5$ for $T_x$. The condition $\Delta T_x < I$ is equivalent to $x^2 + x < 2I$. The largest $x$ satisfying $x^2 + x < 2I = 2(L - E) = 1,400,264$ is $x = 1182 (x = 1183$ for which $x^2 + x = 1,400,672$ is just slightly too big). Thus, every $n \leq N = T_{1183} + L = 275,932,384 + 1,043,999 = 276,976,383$ is a sum of five tetrahedral numbers. This is a substantial improvement over the 250,000,000 obtained previously in [1] from a looser use of the main idea in the above lemma instead of its optimally sharpened formulation given above.

Table I was calculated with a program similar to that employed in [1] to find exceptional numbers with respect to $T_x$. The first run, using 1,000,000 words of memory was done on an IBM 360-75. The print-out was checked by using a different machine, an IBM 360-65, and by varying the code to perform in five groups of 200000 words of memory.

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2. H. E. Salzer & N. Levine, "Table of integers not exceeding 10 00000 that are not expressible as the sum of four tetrahedral numbers," MTAC, v. 12, 1958, pp. 141–144. MR 20 #6194.

* $Q_{z+1}$ may be less than $L$ when $I$ is small. But the result for the case $Q_{z+1} < n < L$ is contained in the hypothesis.