Explicit Inverses and Condition Numbers of Certain Circulants

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Abstract. Explicit inverses and condition numbers of two test-circulants with first rows $\{a, a + h, \cdots, a + (n - 1)h\}$ and $\{a, ah, \cdots, ah^{n-1}\}$ respectively are given in terms of the parameters defining the circulants.

1. Introduction. A circulant with the first row $(1, 2, \cdots, n)$ was used in testing inversion algorithms [6] but its explicit inverse was not given in the list of explicit inverses of some particular matrices in [5]. This paper gives explicit inverses and condition numbers of two circulants whose special cases include the one used in [6].

The simple result that the inverse of a circulant is a circulant [2] can be easily extended to the case of r-circulants. An r-circulant as defined in [4] is a square matrix of order n in which the ith row, $i = 2, 3, \cdots, n$, is obtained from the $(i - 1)$th row by cyclically shifting each element r places to the right. The word "row" can be replaced by the word "column" if "right" is replaced by "down." If we also say that shifting a negative number of places right means shifting left, then an r-circulant is also a $(kn + r)$-circulant, for any integer k.

Theorem 1.1. *The inverse of a nonsingular r-circulant A is an s-circulant B where s satisfies*

$$rs = kn + 1 \tag{1.1}$$

for some integer k.

Proof. Let e_1 denote the first column of I. The set of equations

$$Af = e_1 \tag{1.2}$$

has a unique solution f, since A is nonsingular. Let B be the s-circulant with first column f, where s satisfies (1.1). Then, by Theorem (3.1) in [1], AB is an $(rs = kn + 1)$ circulant with first column e_1; that is, $AB = I$.

2. Explicit Inverses. Since the inverse of a circulant is a circulant we shall hypothesize that the inverse of a circulant which is defined by a few parameters can be explicitly expressed in terms of these parameters. The forms of the expressions can be conveniently observed from the results of numerical experiments on a digital computer and applications of (1.2) then give the required explicit inverses.

In the following two theorems s is defined as in (1.1). A_1 and A_2 are nonsingular r-circulants of order $n \geqq 2$ with first row $\{a, a + h, \cdots, l = a + (n - 1)h\}$ for A_1, and $\{a, ah, \cdots, ah^{n-1}\}$ for A_2. $R_i(A)$ denotes the ith row, $C_i(A)$ the ith column and a_{ij} the (i, j)th element of A.

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THEOREM 2.1. *The inverse of* A_1 *is the s-circulant with the first column* $\{b - \alpha, b, \cdots, b, b + \alpha\}^T$, *where* $b = 2/\{n^2(a + l)\}$ and $\alpha = 1/(nh)$.

Proof. Let y be the first column of an s-circulant B. According to (1.2) B will be the inverse of A if the ith element of

$$A_1 y = R_i(A)C_1(B) = b \sum_{i=1}^{n} [a + (i - 1)h] + \alpha(a_{in} - a_{i1})$$

is e_1. By observation of A_1 we have

$$\begin{aligned} a_{in} - a_{i1} &= (n - 1)h\,, \qquad && i = 1\,, \\ &= -h\,, && i \neq 1\,. \end{aligned}$$

Therefore, $A_1 y = e_1$ if $\alpha nh = 1$ and $bn(a + l)/2 - \alpha h = 0$ which give the required expressions for b and α.

Obviously Theorem 2.1 gives the inverse of a (± 1)-circulant with first row $(1, 2, \cdots, n)$ as the (± 1)-circulant with the first column $n^{-1}(b - 1, b, \cdots, b, b + 1)^T$ where $b = 2/\{n(n + 1)\}$.

THEOREM 2.2. *The inverse of* A_2 *is the s-circulant with first column* $\{b, 0, \cdots, 0, -hb\}^T$ *when* $b = 1/\{a(1 - h^n)\}$.

A proof can be easily constructed similar to that for Theorem 2.1 by using the property of A_2 that

$$\begin{aligned} ha_{in} - a_{i1} &= a(h^n - 1)\,, \qquad && i = 1\,, \\ &= 0\,, && i \neq 1\,. \end{aligned}$$

3. Condition Numbers. Circulants which are usually employed in testing numerical algorithms are the 1-circulant which are generally nonsymmetric and the (-1)-circulants which are always symmetric. One measure of the condition of these matrices, denoted here by $P(A)$, is the ratio of the largest (in modulus) to the smallest eigenvalue [7].

An expression for the eigenvalues of a 1-circulant may be found in [3] and [5]:

$$\lambda_s = \sum_{j=1}^{n} a_j t_s^{\,j-1} \tag{3.1}$$

where $t_s = \cos(2\pi s/n) + i \sin(2\pi s/n)$, $s = 1, 2, \cdots, n$, and a_j is the jth element of the first row of the circulant.

An expression for the n eigenvalues of a (-1)-circulant does not seem to be readily available in the literature but can be easily shown to be

$$\lambda_0;\ \pm\,(\lambda_s \lambda_{n-s})^{1/2}\,, \qquad s = 1, 2, \cdots, (n - 1)/2\,, \tag{3.2}$$

for odd n. The λ_s are the eigenvalues of the 1-circulant whose first row is the same as the (-1)-circulant in question. When n is even the set of n eigenvalues of the (-1)-circulant are $\lambda_{(n/2)}$ and those in (3.3) with $s = 1, 2, \cdots, (n - 2)/2$.

Examinations of the sets of eigenvalues of the (± 1)-circulants of the forms A_1 and A_2 did not reveal expressions for their condition numbers. Since the explicit inverses of A_1 and A_2 as given in Theorems 2.1 and 2.2 are simpler than the matrices themselves, eigenvalues of the inverses were examined. After algebraic and trigonometric manipulations, we have for both the 1-circulants and (-1)-circulants,

$$P(A_1) \sim n + 2a/h\,, \qquad h > 0\,, \tag{3.4}$$

and

$$P(A_2) \sim \text{Max}\,[p_2 = |1 + h|/|1 - h|,\, 1/p_2]\,, \tag{3.5}$$

for large n. The symbol "$\sim$" is read "asymptotically equals." In the special case in which $a = h = 1$ in A_1, (3.4) gives $P(A_1) \sim n$ in agreement with the result in [5].

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