

Explicit Inverses and Condition Numbers of Certain Circulants

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Abstract. Explicit inverses and condition numbers of two test-circulants with first rows $\{a, a + h, \dots, a + (n - 1)h\}$ and $\{a, ah, \dots, ah^{n-1}\}$ respectively are given in terms of the parameters defining the circulants.

1. Introduction. A circulant with the first row $(1, 2, \dots, n)$ was used in testing inversion algorithms [6] but its explicit inverse was not given in the list of explicit inverses of some particular matrices in [5]. This paper gives explicit inverses and condition numbers of two circulants whose special cases include the one used in [6].

The simple result that the inverse of a circulant is a circulant [2] can be easily extended to the case of r -circulants. An r -circulant as defined in [4] is a square matrix of order n in which the i th row, $i = 2, 3, \dots, n$, is obtained from the $(i - 1)$ th row by cyclically shifting each element r places to the right. The word "row" can be replaced by the word "column" if "right" is replaced by "down." If we also say that shifting a negative number of places right means shifting left, then an r -circulant is also a $(kn + r)$ -circulant, for any integer k .

THEOREM 1.1. *The inverse of a nonsingular r -circulant A is an s -circulant B where s satisfies*

$$(1.1) \quad rs = kn + 1$$

for some integer k .

Proof. Let e_1 denote the first column of I . The set of equations

$$(1.2) \quad Af = e_1$$

has a unique solution f , since A is nonsingular. Let B be the s -circulant with first column f , where s satisfies (1.1). Then, by Theorem (3.1) in [1], AB is an $(rs = kn + 1)$ circulant with first column e_1 ; that is, $AB = I$.

2. Explicit Inverses. Since the inverse of a circulant is a circulant we shall hypothesize that the inverse of a circulant which is defined by a few parameters can be explicitly expressed in terms of these parameters. The forms of the expressions can be conveniently observed from the results of numerical experiments on a digital computer and applications of (1.2) then give the required explicit inverses.

In the following two theorems s is defined as in (1.1). A_1 and A_2 are nonsingular r -circulants of order $n \geq 2$ with first row $\{a, a + h, \dots, l = a + (n - 1)h\}$ for A_1 , and $\{a, ah, \dots, ah^{n-1}\}$ for A_2 . $R_i(A)$ denotes the i th row, $C_i(A)$ the i th column and a_{ij} the (i, j) th element of A .

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THEOREM 2.1. *The inverse of A_1 is the s -circulant with the first column $\{b - \alpha, b, \dots, b, b + \alpha\}^T$, where $b = 2/\{n^2(a + l)\}$ and $\alpha = 1/(nh)$.*

Proof. Let y be the first column of an s -circulant B . According to (1.2) B will be the inverse of A if the i th element of

$$A_1y = R_i(A)C_1(B) = b \sum_{i=1}^n [a + (i - 1)h] + \alpha(a_{in} - a_{i1})$$

is e_1 . By observation of A_1 we have

$$\begin{aligned} a_{in} - a_{i1} &= (n - 1)h, & i = 1, \\ &= -h, & i \neq 1. \end{aligned}$$

Therefore, $A_1y = e_1$ if $\alpha nh = 1$ and $bn(a + l)/2 - \alpha h = 0$ which give the required expressions for b and α .

Obviously Theorem 2.1 gives the inverse of a (± 1) -circulant with first row $(1, 2, \dots, n)$ as the (± 1) -circulant with the first column $n^{-1}(b - 1, b, \dots, b, b + 1)^T$ where $b = 2/\{n(n + 1)\}$.

THEOREM 2.2. *The inverse of A_2 is the s -circulant with first column $\{b, 0, \dots, 0, -hb\}^T$ when $b = 1/\{a(1 - h^n)\}$.*

A proof can be easily constructed similar to that for Theorem 2.1 by using the property of A_2 that

$$\begin{aligned} ha_{in} - a_{i1} &= a(h^n - 1), & i = 1, \\ &= 0, & i \neq 1. \end{aligned}$$

3. Condition Numbers. Circulants which are usually employed in testing numerical algorithms are the 1-circulant which are generally nonsymmetric and the (-1) -circulants which are always symmetric. One measure of the condition of these matrices, denoted here by $P(A)$, is the ratio of the largest (in modulus) to the smallest eigenvalue [7].

An expression for the eigenvalues of a 1-circulant may be found in [3] and [5]:

$$(3.1) \quad \lambda_s = \sum_{j=1}^n a_j t_s^{j-1}$$

where $t_s = \cos(2\pi s/n) + i \sin(2\pi s/n)$, $s = 1, 2, \dots, n$, and a_j is the j th element of the first row of the circulant.

An expression for the n eigenvalues of a (-1) -circulant does not seem to be readily available in the literature but can be easily shown to be

$$(3.2) \quad \lambda_0; \pm (\lambda_s \lambda_{n-s})^{1/2}, \quad s = 1, 2, \dots, (n - 1)/2,$$

for odd n . The λ_s are the eigenvalues of the 1-circulant whose first row is the same as the (-1) -circulant in question. When n is even the set of n eigenvalues of the (-1) -circulant are $\lambda_{(n/2)}$ and those in (3.3) with $s = 1, 2, \dots, (n - 2)/2$.

Examinations of the sets of eigenvalues of the (± 1) -circulants of the forms A_1 and A_2 did not reveal expressions for their condition numbers. Since the explicit inverses of A_1 and A_2 as given in Theorems 2.1 and 2.2 are simpler than the matrices themselves, eigenvalues of the inverses were examined. After algebraic and

trigonometric manipulations, we have for both the 1-circulants and (-1) -circulants,

$$(3.4) \quad P(A_1) \sim n + 2a/h, \quad h > 0,$$

and

$$(3.5) \quad P(A_2) \sim \text{Max} [p_2 = |1 + h|/|1 - h|, 1/p_2],$$

for large n . The symbol " \sim " is read "asymptotically equals." In the special case in which $a = h = 1$ in A_1 , (3.4) gives $P(A_1) \sim n$ in agreement with the result in [5].

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