Chebyshev Approximations for the Fresnel Integrals*

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Abstract. Rational Chebyshev approximations have been computed for the Fresnel integrals $C(x)$ and $S(x)$ for arguments in the intervals [0., 1.2] and [1.2, 1.6], and for the related functions $f(x)$ and $g(x)$ for the intervals [1.6, 1.9], [1.9, 2.4] and [2.4, ∞]. Maximal relative errors range down to 2×10^{-19}.

1. Introduction. The Fresnel integrals are defined [1] by

$$C(x) - iS(x) = \int_0^x i^{-t^2} dt = \int_0^x \exp\,(-i\pi t^2/2) dt\,. \tag{1}$$

For small arguments the usual Taylor series expansions are quite useful computationally, while for large arguments the forms

$$C(x) = \tfrac{1}{2} + f(x) \sin\,(\pi x^2/2) - g(x) \cos\,(\pi x^2/2) \tag{2a}$$

and

$$S(x) = \tfrac{1}{2} - f(x) \cos\,(\pi x^2/2) - g(x) \sin\,(\pi x^2/2)\,, \tag{2b}$$

where $f(x)$ and $g(x)$ have well-known asymptotic expansions, are most useful. For values of x greater than that corresponding to the first maximum function value (in the vicinity of $x = 1$), evaluation of the Taylor series is subject to loss of accuracy through subtraction error. Since the asymptotic forms are not useful for small $|x|$, there is a region for which accurate evaluation of the Fresnel integrals is difficult.

In recent years a number of papers presenting approximations have appeared [2]–[4]. All of the approximations given are of somewhat limited usefulness, however. The Chebyshev series expansions given by Németh [4] converge painfully slowly, while the single approximation given by Boersma [3] is of limited accuracy and subject to subtraction error during evaluation. The approximation forms used by Syrett and Wilson [2] are generally quite inefficient and involve awkward transformations of variable. None of the previous investigators has considered approximation by rational functions, although such approximations are generally more efficient than pure polynomial approximations. It is our purpose to present efficient rational approximations for $S(x)$ and $C(x)$ when $|x|$ is small, and for $f(x)$ and $g(x)$ for all other x, with maximal relative errors ranging down to 10^{-19} in some cases.

2. Approximation Forms. The approximation forms and intervals used are:

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$$C_{lm}(x) = xR_{lm}(x^4) \quad S_{lm}(x) = x^3R_{lm}(x^4) \quad \text{for} \quad |x| \leqq 1.2 \text{ and } 1.2 \leqq |x| \leqq 1.6 ; \tag{3}$$

$$f_{lm} = x^{-1}R_{lm}(x^{-4}) \quad g_{lm} = x^{-3}R_{lm}(x^{-4}) \quad \text{for} \quad 1.6 \leqq |x| \leqq 1.9 \text{ and } 1.9 \leqq |x| \leqq 2.4 ; \tag{4}$$

and

$$f_{lm}(x) = x^{-1}\{1/\pi + x^{-4}R_{lm}(x^{-4})\} \quad g_{lm}(x) = x^{-3}\{1/\pi^2 + x^{-4}R_{lm}(x^{-4})\} \quad \text{for} \quad 2.4 \leqq |x| , \tag{5}$$

where R_{lm} is a rational function of degree l in the numerator and m in the denominator. The forms (3) and (5) are based upon the Taylor series and asymptotic forms of the functions involved, while the forms (4) are based upon the results of much experimentation.

The choice of the intervals of approximation again resulted from experimentation. While they are not optimal in this respect, the intervals were chosen so that a given choice of degree of numerator and denominator would result in roughly the same accuracy for each interval.

3. Computations. All computations were carried out in 25-significant figure arithmetic on a CDC 3600 computer. The approximations were computed using the Remes algorithm for rational Chebyshev approximations [5], [6]. Because the approximation forms used for large $|x|$ correctly emulate the asymptotic behaviour of $f(x)$ and $g(x)$, the error of approximation vanishes as $|x| \to \infty$. Consequently these approximations could be computed for large (but finite) upper limits to the interval of approximation.

Function values were computed as needed using the Taylor series expansions of $C(x)/x$ and $S(x)/x^3$ for $|x| \leqq 2.5$, and using the most accurate approximations given by Syrett and Wilson [2] for $2.5 \leqq |x| \leqq 4.0$. For $|x| > 4.0$ the asymptotic expansions for $f(x)$ and $g(x)$ were converted into continued fractions by means of the QD algorithm [7]. The function routines were extensively checked against the excellent tables of Syrett and Wilson, and against each other in slightly overlapping regions. These tests indicated an accuracy of 20S in the master functions.

The relative error curves

$$\delta_{lm}(x) = (A(x) - A_{lm}(x))/A(x) \tag{6}$$

where A refers to C, S, f, or g, were all levelled to three significant figures. In addition each approximation, with the coefficients rounded as they appear in the tables, was tested against the master routines for 5000 pseudo-random arguments. In all cases maximal errors agreed (within roundoff) in magnitude and location with those given by the error curves (6) in the Remes algorithm.

4. Results. All results are given in tabular form in the microfiche supplement to this issue of the journal. Tables I–IV list the values of

$$E_{lm} = -100 \log_{10} \max |\delta_{lm}|$$

where the maximum is taken over the appropriate interval, for the initial segments of the various L_∞ Walsh arrays. An examination of the tables indicates E_{lm} is generally quite close to maximal for fixed $l + m$ along the line $l = m$. Tables V–VIII present the coefficients for the cases $l = m$. All coefficients are given to an accuracy greater than that justified by the maximal errors, but reasonable additional rounding should not greatly affect the overall accuracies.

Not all of the approximations have been checked for numerical stability of evaluation, but most of those that were checked proved to be quite stable numerically when the numerator and denominator polynomials were evaluated by nested multiplication. The few exceptions all occurred for the approximations for $S(x)$ and $C(x)$ over the interval $1.2 \leqq |x| \leqq 1.6$, when large subtraction errors occurred. Transforming the numerator and denominator polynomials into their equivalent finite Chebyshev polynomial expansions and using the Clenshaw-Rice scheme [8] for evaluating these gave great numerical stability with the usual penalty in speed of evaluation.

For $|x| \geqq 1.6$, Eqs. (2) must be used to compute $S(x)$ and $C(x)$. If we denote by $\Delta\beta$ the absolute error in the quantity β, and by $\delta\beta$ the relative error, i.e., $\delta\beta = \Delta\beta/\beta$, then we find from (2)

$$\delta C(x) \approx \frac{f(x) \sin(u)}{C(x)} \delta f(x) - \frac{g(x) \cos(u)}{C(x)} \delta g(x) + [f(x) \cos(u) + g(x) \sin(u)] \frac{\pi x}{C(x)} \Delta x ,$$

where $u = \pi x^2/2$. If we assume $\Delta x = 0$, we find the direct contribution of $|\delta f(x)|$ and $|\delta g(x)|$ to $|\delta C(x)|$ is

$$|\delta C(x)| \leqq \frac{\max |f(x)|}{\min |C(x)|} |\delta f(x)| + \frac{\max |g(x)|}{\min |C(x)|} |\delta g(x)| = B|\delta f(x)| + C|\delta g(x)| .$$

Similarly,

$$|\delta S(x)| \leqq \frac{\max |f(x)|}{\min |S(x)|} |\delta f(x)| + \frac{\max |g(x)|}{\min |S(x)|} |\delta g(x)| = D|\delta f(x)| + E|\delta g(x)| .$$

Using the tables in [1] rough upper bounds on B, C, D, and E are easily computed for the intervals of interest to us, and are presented in Table IX. These bounds will be useful in deciding upon the accuracies necessary for the approximations for $f(x)$ and $g(x)$ in order to obtain a desired accuracy in the computation of $C(x)$ or $S(x)$. We note in particular that generally an error in $g(x)$ ten times as large as that in $f(x)$ can be tolerated for a given computation of $C(x)$ or $S(x)$.

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