Approximations for Elliptic Integrals*

By Yudell L. Luke

Abstract. Closed-form approximations are derived for the three kinds of incomplete elliptic integrals by using the Padé approximations for the square root. An effective analytical representation of the error is presented. Approximations for the complete integrals based on trapezoidal-type integration formulae are also developed.

1. Approximations for the Square Root. We start with the following elementary identities:

$$(1)\qquad \tanh\theta = \left\{\frac{\cosh(2n+1)\theta}{\cosh\theta}\right\} \Big/ \left\{\frac{\sinh(2n+1)\theta}{\sinh\theta}\right\} - \frac{e^{-(2n+1)\theta}\tanh\theta}{\sinh(2n+1)\theta},$$

$$(2)\qquad \coth\theta = \left\{\frac{\sinh(2n+1)\theta}{\sinh\theta}\right\} \Big/ \left\{\frac{\cosh(2n+1)\theta}{\cosh\theta}\right\} + \frac{e^{-(2n+1)\theta}\coth\theta}{\cosh(2n+1)\theta},$$

$$(3)\qquad \tanh\theta = \left\{\frac{\sinh\theta\sinh 2n\theta}{\cosh\theta}\right\} \Big/ \cosh 2n\theta + \frac{e^{-2n\theta}\tanh\theta}{\cosh 2n\theta},$$

$$(4)\qquad \coth\theta = \{\cosh 2n\theta\} \Big/ \left\{\frac{\sinh\theta\sinh 2n\theta}{\cosh\theta}\right\} - \frac{e^{-2n\theta}\coth\theta}{\sinh 2n\theta}.$$

Now

$$\frac{\cosh(2n+1)\theta}{\cosh\theta}, \qquad \frac{\sinh(2n+1)\theta}{\sinh\theta}, \qquad \cosh 2n\theta \quad\text{and}\quad \frac{\sinh\theta\sinh 2n\theta}{\cosh\theta}$$

are polynomials in $\sinh^2\theta$ of degree n. So if $z = \sinh^2\theta$, then $\tanh\theta = (1 + 1/z)^{-1/2}$ and clearly (1), (3) and (2), (4) give rational approximations for $(1 + 1/z)^{-1/2}$ and $(1 + 1/z)^{1/2}$, respectively. In the above remainder terms, we take $e^{\phi} = (1 + z)^{1/2} \pm z^{1/2}$, where the sign is chosen so that $|e^{\phi}| > 1$. This is possible for all z except $-1 \leqq z \leqq 0$. For z fixed, $z \neq 0$, $z \neq -1$, $|\arg(1 + 1/z)| < \pi$, the remainder terms $\to 0$ as $n \to \infty$.

From [1], we see that the approximations (1)–(4) are the (n, n), (n, n), $(n - 1, n)$ and $(n + 1, n)$ approximants of the Padé matrix table. We note that the polynomials in these approximations can be expressed in terms of the Chebyshev polynomials, a fact which has also been observed by Longman [2].

The rational approximations can be easily decomposed into a sum of partial fractions. For our immediate applications we use the representations (1), (2) in a slightly different form:

Received February 12, 1968.

* This research was sponsored by the Atomic Energy Commission under Contract No. AT(11-1)-1619.

$$(1 - z)^{-1/2} = (2n + 1)^{-1}\left[1 + 2\sum_{m=1}^{n} (1 - z\sin^2\theta_m)^{-1}\right] + V_n(z) \tag{5}$$

$$(1 - z)^{1/2} = 1 - 2z(2n + 1)^{-1}\sum_{m=1}^{n} \frac{\sin^2\theta_m}{1 - z\cos^2\theta_m} + W_n(z)\,, \tag{6}$$

$$\theta_m = \frac{m\pi}{2n + 1}\,. \tag{7}$$

To describe the remainder terms, we have need for the following definition. Let

$$e^{\zeta} = \frac{2 - z \pm 2(1 - z)^{1/2}}{z}\,, \tag{8}$$

where the sign is chosen so that $|e^{\zeta}|$ lies outside the unit circle which is possible for all z except $z \geqq 1$. Then

$$V_n(z) = \frac{4e^{-(2n+3/2)\zeta}}{z^{1/2}(1 - e^{-\zeta})[1 + e^{-(2n+1)\zeta}]}\,, \tag{9}$$

$$W_n(z) = -\frac{z^{1/2}(1 - e^{-\zeta})e^{-(2n+1/2)\zeta}}{[1 - e^{-(2n+1)\zeta}]}\,, \tag{10}$$

and for z fixed,

$$\lim_{n\to\infty} \{V_n(z) \text{ and } W_n(z)\} = 0\,, \qquad z \neq 1\,,\ |\arg(1 - z)| < \pi\,. \tag{11}$$

2. Approximations for the Incomplete Elliptic Integrals of the First and Third Kinds. We write

$$\begin{aligned} F(\phi, k, \nu) &= \int_0^{\phi} (1 - \nu^2\sin^2\alpha)^{-1}(1 - k^2\sin^2\alpha)^{-1/2}d\alpha\,, \\ &1 - \nu^2\sin^2\phi \neq 0\,, |\arg(1 - k^2)| < \pi\,, |\arg(1 - k^2\sin^2\phi)| < \pi\,, \end{aligned} \tag{12}$$

which is the incomplete elliptic integral of the third kind if $\nu \neq 0$. If $(1 - \nu^2\sin^2\phi) < 0$, we interpret the integral in the Cauchy sense. When $\nu = 0$, we have the incomplete elliptic integral of the first kind which is usually notated as $F(\phi, k)$. Expressions for analytic continuation of the elliptic integrals are detailed in [3]. Now

$$\begin{aligned} A(\phi, \nu) = F(\phi, 0, \nu) &= (1 - \nu^2)^{-1/2}\arctan[(1 - \nu^2)^{1/2}\tan\phi] \\ &\qquad\qquad \text{if } \nu^2 \neq 1\,, |\arg(1 - \nu^2)| < \pi \\ &= \tfrac{1}{2}(\nu^2 - 1)^{1/2}\ln\left|\frac{1 + (\nu^2 - 1)^{1/2}\tan\phi}{1 - (\nu^2 - 1)^{1/2}\tan\phi}\right|, \qquad \phi \text{ real}\,, \text{if } \nu^2 > 1\,, \\ &= \tan\phi \quad \text{if} \quad \nu^2 = 1\,. \end{aligned} \tag{13}$$

Here arc tan x is given its principal value, that is, $-\pi/2 < \arctan x < \pi/2$ for $-\infty < x < \infty$. Then with the aid of (5) and with the same restrictions as in (12), we get

$$F(\phi, k, \nu) = F_n(\phi, k, \nu) + Q_n(\phi, k, \nu)\ ,$$

(14)
$$F_n(\phi, k, \nu) = (2n+1)^{-1}\left[A(\phi, \nu)\left\{1 - 2\nu^2 \sum_{m=1}^{n} (k^2 \sin^2 \theta_m - \nu^2)^{-1}\right\} + 2k^2 \sum_{m=1}^{n} \frac{\sin^2 \theta_m \arctan(\sigma_m \tan \phi)}{\sigma_m(k^2 \sin^2 \theta_m - \nu^2)}\right],$$

$$\theta_m = \frac{m\pi}{2n+1}, \sigma_m = (1 - k^2 \sin^2 \theta_m)^{1/2},$$

(15)
$$F_n(\pi/2, k, \nu) = \frac{\pi}{2(2n+1)}\left[(2/\pi)A(\pi/2, \nu)\left\{1 - 2\nu^2 \sum_{m=1}^{n} (k^2 \sin^2 \theta_m - \nu^2)^{-1}\right\} + 2k^2 \sum_{m=1}^{n} \frac{\sin^2 \theta_m}{\sigma_m(k^2 \sin^2 \theta_m - \nu^2)}\right],$$

$$(2/\pi)A(\pi/2, \nu) = (1 - \nu^2)^{-1/2} \quad \text{if} \quad \nu^2 \neq 1\ , |\arg(1 - \nu^2)| < \pi\ ,$$
$$= 0 \quad \text{if} \quad \nu^2 > 1\ .$$

(16)
$$Q_n(\phi, k, \nu) = \int_0^{\phi} \frac{V_n(k^2 \sin^2 \alpha)}{1 - \nu^2 \sin^2 \alpha}\, d\alpha\ .$$

We shall develop a very convenient asymptotic formula for $Q_n(\phi, k, \nu)$, but first we remark that in the application of (14), (15), it might happen that for a particular set of parameters, $k^2 \sin^2 \theta_m = \nu^2$ in which event limiting forms in these equations must be taken. This can always be avoided by a proper choice of n.

Next we turn to an analysis of the error. Suppose temporarily that $0 \leqq k^2 < 1$ and $-\pi/2 < \phi < \pi/2$. Then from (8)

(17)
$$z = k^2 \sin^2 \alpha = 2(1 + \cosh \zeta)^{-1}\ ,$$
$$d\alpha = -\frac{z^{1/2}(1 - z)^{1/2}}{2(k^2 - z)^{1/2}}\, d\zeta\ .$$

It readily follows that

(18)
$$Q_n(\phi, k, \nu) = 2\int_{\eta}^{\infty} \frac{e^{-(2n+3/2)\zeta}\{\sinh \zeta/(1 + \cosh \zeta)\}d\zeta}{(1 + e^{-\zeta})(1 + e^{-(2n+1)\zeta})\left(1 - \dfrac{2\nu^2}{k^2(1 + \cosh \zeta)}\right)\left(k^2 - \dfrac{2}{1 + \cosh \zeta}\right)^{1/2}},$$
$$\eta = \operatorname{arc\,cosh}\left(\frac{2}{k^2 \sin^2 \phi} - 1\right).$$

Let $\zeta = \eta + y$. Then (18) goes into a form to which Watson's lemma is applicable. We find

(19)
$$Q_n(\phi, k, \nu) = \frac{2 \tan \phi \left(\dfrac{1 - \delta}{k \sin \phi}\right)^{4n+2}}{(1 - \nu^2 \sin^2 \phi)(4n + 3)} \times \left[1 + \frac{\left\{1 - \dfrac{\delta}{\cos^2 \phi} - \dfrac{2\nu^2 \delta \sin^2 \phi}{1 - \nu^2 \sin^2 \phi}\right\}}{4n + 3} + O(n^{-2})\right] + \mu_n$$

$$\mu_n = O\left(\frac{1-\delta}{k \sin \phi}\right)^{8n}, \qquad \phi \neq \pi/2\,, \delta = (1 - k^2 \sin^2 \phi)^{1/2}\,,$$

$$(20) \qquad Q_n(\pi/2, k, \nu) = \frac{\left(\dfrac{2\pi}{4n+3}\right)^{1/2}\left(\dfrac{1-\delta}{k}\right)^{4n+2}}{(1-\nu^2)\delta^{1/2}} \times \left[1 - \frac{\left\{\dfrac{\delta^2 - 4\delta + 1}{8\delta} + \dfrac{\nu^2\delta}{1-\nu^2}\right\}}{4n+3} + O(n^{-2})\right] + \mu_n\,,$$

δ and μ_n as in (19) with $\phi = \pi/2$. The results (19), (20) are valid for complex values of $k^2 \sin^2 \phi$ provided $|\arg(1 - k^2)| < \pi$, $|\arg(1 - k^2 \sin^2 \phi)| < \pi$ and $(1 - \delta)/k \sin \phi$ is replaced by $(1 \pm \delta)/k \sin \phi$ where the sign is chosen so that $|(1 \pm \delta)/k \sin \phi| < 1$. Clearly, for ϕ, k and ν fixed,

$$(21) \qquad \lim_{n\to\infty} Q_n(\phi, k, \nu) = 0\,, \qquad 1 - \nu^2 \sin^2 \phi \neq 0\,,$$
$$|\arg(1 - k^2)| < \pi\,, |\arg(1 - k^2 \sin^2 \phi)| < \pi\,.$$

The following results illustrate the approximations and the striking realism of the formulas for the error. The column "Error" means the true error, and the column "Approximate Error, (19)", for example, means the value of $Q_n(\phi, k, \nu)$ as given by (19) with $O(n^{-2})$ and μ_n omitted. The number in parentheses following a base number indicates the power of 10 by which the base number must be multiplied. Analogous descriptors like those above are used for all other tables in this paper.

n	$F_n(\pi/4, 3^{1/2}/2, 2^{-1/2})$	*Error*	*Approximate Error*, (19)
1	0.94884 29702 686	0.543(−3)	0.513(−3)
2	0.94938 05802 787	0.489(−5)	0.477(−5)
3	0.94938 54233 614	0.500(−7)	0.493(−7)
4	0.94938 54728 231	0.546(−9)	0.541(−9)
5	0.94938 54733 633	0.63(−11)	0.62(−11)
6	0.94938 54733 694	0.1(−12)	0.73(−13)

n	$F_n(\pi/4, 1, 2^{-1/2})$	*Error*	*Approximate Error*, (19)
1	0.98417 06846 361	0.174(−2)	0.168(−2)
2	0.98587 72891 996	0.337(−4)	0.331(−4)
3	0.98591 02359 695	0.739(−6)	0.732(−6)
4	0.98591 09574 866	0.173(−7)	0.172(−7)
5	0.98591 09744 025	0.425(−9)	0.423(−9)
6	0.98591 09748 163	0.107(−10)	0.107(−10)
8	0.98591 09748 270	—	0.72(−14)

n	$F_n(\pi/4, 3^{1/2}/2, 0)$	*Error*	*Approximate Error*, (19)
1	0.85079 29130 329	0.431(−3)	0.419(−3)
2	0.85121 99306 546	0.382(−5)	0.377(−5)
3	0.85122 37104 060	0.387(−7)	0.384(−7)
4	0.85122 37486 511	0.420(−9)	0.418(−9)
5	0.85122 37490 664	0.48(−11)	0.48(−11)
6	0.85122 37490 711	0.1(−12)	0.56(−13)
8	0.85122 37490 712	—	0.81(−17)

n	$F_n(\pi/2, 3^{1/2}/2, 2^{-1/2})$	*Error*	*Approximate Error*, (20)
1	3.14171 75952 888	0.931(−1)	0.948(−1)
2	3.22622 30627 726	0.855(−2)	0.855(−2)
3	3.23395 26637 116	0.821(−3)	0.820(−3)
4	3.23469 21065 800	0.814(−4)	0.813(−4)
5	3.23476 52331 988	0.824(−5)	0.823(−5)
6	3.23477 26248 822	0.846(−6)	0.846(−6)
8	3.23477 34620 495	0.920(−8)	0.920(−8)
10	3.23477 34711 468	0.103(−9)	0.103(−9)
15	3.23477 34712 495	—	0.14(−14)

3. Approximations for the Incomplete Elliptic Integral of the Second Kind. Let

$$E(\phi, k) = \int_0^\phi (1 - k^2 \sin^2 \alpha)^{1/2} d\alpha , \qquad |\arg (1 - k^2)| < \pi , |\arg (1 - k^2 \sin^2 \phi)| < \pi . \tag{22}$$

Using (6) and the fact that

$$\int_0^\phi \frac{\sin^2 \alpha \, d\alpha}{1 - a^2 \sin^2 \alpha} = a^{-2}\{(1 - a^2)^{-1/2} \arc\tan [(1 - a^2)^{1/2} \tan \phi] - \phi\} , \qquad a^2 \neq 1 , |\arg (1 - a^2)| < \pi , \tag{23}$$

we find that under the same restrictions as for (22),

$$E(\phi, k) = E_n(\phi, k) + S_n(\phi, k) , \tag{24}$$

$$E_n(\phi, k) = (2n + 1)\phi - 2(2n + 1)^{-1} \sum_{m=1}^{n} \frac{\tan^2 \theta_m \arc\tan (\rho_m \tan \phi)}{\rho_m} , \qquad \theta_m = \frac{m\pi}{2n + 1} , \rho_m = (1 - k^2 \cos^2 \theta_m)^{1/2} , \tag{25}$$

$$E_n(\pi/2, k) = (\pi/2)\left[(2n + 1) - 2(2n + 1)^{-1} \sum_{m=1}^{n} \frac{\tan^2 \theta_m}{\rho_m} \right] , \tag{26}$$

$$S_n(\phi, k) = \int_0^\phi W_n(k^2 \sin^2 \alpha) d\alpha . \tag{27}$$

In the above arc tan is evaluated as in the discussion following (13). After the manner of getting (19), (20) we have

$$\begin{aligned} S_n(\phi, k) &= - \frac{2\delta^2 \tan \phi}{(4n + 1)} \left(\frac{1 \pm \delta}{k \sin \phi} \right)^{4n+2} \\ &\quad \times \left[1 - \frac{\left\{ \dfrac{3\delta^2 + \delta - 2}{\delta} - \delta \tan^2 \phi \right\}}{4n + 1} + O(n^{-2}) \right] + \rho_n , \\ \rho_n &= O\left(\frac{1 \pm \delta}{k \sin \phi} \right)^{8n} , \phi \neq \pi/2 , \delta = (1 - k^2 \sin^2 \phi)^{1/2} , \end{aligned} \tag{28}$$

$$\begin{aligned} S_n(\pi/2, k) &= - \left(\frac{2\pi}{4n + 1} \right)^{1/2} \left(\frac{1 \pm \delta}{k} \right)^{4n+2} \\ &\quad \times \delta^{3/2} \left[1 - \frac{(9\delta^2 + 4\delta - 7)}{8\delta(4n + 1)} + O(n^{-2}) \right] + \rho_n , \end{aligned} \tag{29}$$

δ and ρ_n as in (28) with $\phi = \pi/2$. In (28), (29), we impose the same conditions as in (22). Also the sign in $(1 \pm \delta)/k \sin \phi$ is chosen as in the discussion following (20). For ϕ and k fixed,

$$\lim_{n\to\infty} S_n(\phi, k) = 0\,, |\arg (1 - k^2)| < \pi\,, |\arg (1 - k^2 \sin^2 \phi)| < \pi\,. \tag{30}$$

Two sample calculations follow.

n	$E_n(\pi/4, 3^{1/2}/2)$	*Error*	*Approximate Error*, (28)
1	0.72852 36735 11	−0.300(−3)	−0.269(−3)
2	0.72822 67215 94	−0.257(−5)	−0.249(−5)
3	0.72822 41810 03	−0.255(−7)	−0.252(−7)
4	0.72822 41557 32	−0.275(−9)	−0.272(−9)
5	0.72822 41554 60	−0.3(−11)	−0.308(−11)
6	0.72822 41554 57	—	−0.358(−13)

n	$E_n(\pi/2, 3^{1/2}/2)$	*Error*	*Approximate Error*, (29)
1	1.22710 48575 73	−0.160(−1)	−0.167(−1)
2	1.21233 86155 36	−0.128(−2)	−0.131(−2)
3	1.21117 30503 58	−0.117(−3)	−0.118(−3)
4	1.21106 73123 55	−0.113(−4)	−0.114(−4)
5	1.21105 71499 11	−0.112(−5)	−0.113(−5)
6	1.21105 61414 31	−0.114(−6)	−0.114(−6)
8	1.21105 60287 86	−0.122(−8)	−0.122(−8)
10	1.21105 60275 82	−0.13(−10)	−0.135(−10)
15	1.21105 60275 69	—	−0.186(−15)

We remark that further representations for the incomplete elliptic integrals can be developed using (3), (4) and also (1)–(4) together with the fact that $y = y^2y^{-1}$, $y^2 = 1 - z$, but we omit such considerations.

4. Further Approximations for the Complete Elliptic Integrals. Consider the more general integral,

$$\begin{gathered} B(k, \nu, \omega) = \int_0^{\pi/2} (1 - \nu^2 \sin^2 t)^{-1}(1 - k^2 \sin^2 t)^{-\omega} dt\,, \\ \nu^2 < 1\,, \omega < 1\,, |\arg (1 - k^2)| < \pi\,. \end{gathered} \tag{31}$$

Using trapezoidal-type numerical integration formulas discussed in a previous paper [4], we have

$$B(k, \nu, \omega) = B_n(k, \nu, \omega) + G_n(k, \nu, \omega)\,, \tag{32}$$

$$B(k, \nu, \omega) = C_n(k, \nu, \omega) + H_n(k, \nu, \omega)\,, \tag{33}$$

$$\begin{aligned} B_n(k, \nu, \omega) = \frac{\pi}{2n} \Bigg[& \sum_{m=0}^{n} (1 - \nu^2 \sin^2 \beta_m)^{-1}(1 - k^2 \sin^2 \beta_m)^{-\omega} \\ & - \tfrac{1}{2} - \tfrac{1}{2}(1 - \nu^2)^{-1}(1 - k^2)^{-\omega} \Bigg], \qquad \beta_m = \frac{m\pi}{2n}\,, \end{aligned} \tag{34}$$

$$\begin{aligned} C_n(k, \nu, \omega) = \frac{\pi}{2n} \sum_{m=1}^{n} (1 - \nu^2 \sin^2 \phi_m)^{-1}(1 - k^2 \sin^2 \phi_m)^{-\omega}\,, \\ \phi_m = \frac{(2m - 1)\pi}{4n} \end{aligned} \tag{35}$$

$$(36)\qquad \begin{aligned} G_n(k, \nu, \omega) &= -2\sum_{r=1}^{\infty} L_{rn}(k, \nu, \omega)\,, \\ H_n(k, \nu, \omega) &= -2\sum_{r=1}^{\infty} (-)^r L_{rn}(k, \nu, \omega)\,, \end{aligned}$$

$$(37)\qquad L_n(k, \nu, \omega) = \int_0^{\pi/2} \frac{\cos 4nt\, dt}{(1-\nu^2\sin^2 t)(1-k^2\sin^2 t)^{\omega}}\,.$$

We now get an asymptotic formula for the evaluation of $L_n(k, \nu, \omega)$. It is convenient to consider the contour integral

$$(38)\qquad M_n^*(k, \nu, \omega) = \int_C \frac{e^{2int}dt}{(1-\nu^2\sin^2 t)(1-k^2\sin^2 t)^{\omega}}$$

under the temporary assumptions that $0 \leqq k^2 < 1$, $\nu^2 < 1$ and $\omega < 1$, as in (31), where C is a rectangle with vertices at $(0, 0)$, $(\pi/2, 0)$, $(\pi/2, R)$ and $(0, R)$ with $R >$ arc cosh k^{-1}. Let $R \to \infty$ and so obtain

$$(39)\qquad \begin{aligned} M_n(k, \nu, \omega) &= \int_0^{\pi/2} \frac{\cos 2nt\, dt}{(1-\nu^2\sin^2 t)(1-k^2\sin^2 t)^{\omega}} \\ &= (-)^n \sin \pi\omega \int_{\operatorname{arc\,cosh} k^{-1}}^{\infty} \frac{e^{-2ny}dy}{(1-\nu^2\cosh^2 y)(k^2\cosh^2 y - 1)^{\omega}}\,. \end{aligned}$$

Put $y =$ arc cosh $k^{-1} + x$ and upon application of Watson's lemma to the resulting integral, we get

$$(40)\qquad \begin{aligned} M_n(k, \nu, \omega) &= \frac{(-)^n(\pi/2)k^2 n^{\omega-1}}{\Gamma(\omega)(k^2-\nu^2)\delta^{\omega}}\left(\frac{1-\delta}{k}\right)^{2n} \\ &\quad\times\left[1 - \frac{(1-\omega)}{4n}\left\{\frac{\omega(\delta^2+1)}{2\delta} - \frac{2\nu^2\delta}{k^2-\nu^2}\right\} + O(n^{-2})\right], \\ &\qquad k^2 \neq \nu^2\,,\ \delta = (1-k^2)^{1/2}\,, \end{aligned}$$

$$(41)\qquad \begin{aligned} M_n(k, \nu, \omega) &= \frac{(-)^n(\pi/2)n^{\omega}}{\Gamma(\omega+1)\delta^{\omega+1}}\left(\frac{1-\delta}{k}\right)^{2n}\left[1 + \frac{\omega(\omega+1)(\delta^2+1)}{4n\delta} + O(n^{-2})\right], \\ &\qquad k^2 = \nu^2\,,\ \delta \text{ as in } (40)\,. \end{aligned}$$

Notice that $L_n(k, \nu, \omega) = M_{2n}(k, \nu, \omega)$. Also the series expansions for $G_n(k, \nu, \omega)$ and $H_n(k, \nu, \omega)$, see (36), are usually so rapidly convergent that they can be well approximated by use of the first term only. To extend the results to complex k, we require $|\arg(1 - k^2)| < \pi$, and in (40), (41) replace $(1 - \delta)/k$ by $(1 \pm \delta)/k$ and choose the sign so that $|(1 \pm \delta)/k| < 1$. Hence

$$(42)\qquad \begin{aligned} &\lim_{n\to\infty} \{G_n(k, \nu, \omega) \ \text{ and } \ H_n(k, \nu, \omega)\} = 0\,, \\ &\qquad \nu^2 < 1\,, \omega < 1\,, k^2 \neq 1\,, |\arg(1-k^2)| < \pi\,. \end{aligned}$$

The foregoing results are also valid for all ν, $\nu^2 \pm 1$, and all ω, $R(\omega) < 1$, $-\pi < \arg\omega \leqq \pi$. When $\nu^2 > 1$, we interpret (31) in the Cauchy sense.

Some sample calculations based on (35) follow. The column "Approximate Error" is the value of $2L_n(k, \nu, \omega) = 2M_{2n}(k, \nu, \omega)$ as given by (40), (41) without the $O(n^{-2})$ term. It calls for remark that for virtually the same number of terms, there is little difference in the accuracy of the formulas developed in this section and the corresponding formulas in the previous sections. However, (26) must be used with some caution because differences of large numbers occur since for m large, $m \leqq n$, θ_m is near $\pi/2$ and $\tan^2 \theta_m$ is large.

n	$C_n(3^{1/2}/2, 2^{-1/2}, \frac{1}{2})$	*Error*	*Approximate Error*
1	2.64922 35375 456	0.586(0)	0.642(0)
2	3.18230 94294 968	0.525(−1)	0.484(−1)
3	3.23012 37523 606	0.465(−2)	0.433(−2)
4	3.23433 87288 215	0.435(−3)	0.414(−3)
5	3.23473 10853 791	0.424(−4)	0.410(−4)
6	3.23476 92226 923	0.425(−5)	0.414(−5)
8	3.23477 34263 973	0.449(−7)	0.441(−7)
10	3.23477 34707 574	0.492(−9)	0.486(−9)
15	3.23477 34712 495	—	0.671(−14)

n	$C_n(3^{1/2}/2, 0, -\frac{1}{2})$	*Error*	*Approximate Error*
1	1.24182 35332 245	−0.308(−1)	−0.275(−1)
2	1.21214 27083 881	−0.109(−2)	−0.102(−2)
3	1.21111 93193 818	−0.633(−4)	−0.608(−4)
4	1.21106 05058 942	−0.448(−5)	−0.434(−5)
5	1.21105 63793 937	−0.352(−6)	−0.343(−6)
6	1.21105 60570 715	−0.295(−7)	−0.289(−7)
8	1.21105 60278 027	−0.234(−9)	−0.231(−9)
10	1.21105 60275 705	−0.21 (−11)	−0.203(−11)
15	1.21105 60275 684	—	−0.187(−16)

5. Acknowledgement. I am indebted to Miss Rosemary Moran for the numerical calculations.

Midwest Research Institute
Kansas City, Missouri 64110

1. Y. L. Luke, "The Padé table and the τ-method," *J. Math. Phys.*, v. 37, 1958, pp. 110–127. MR **20** #5558.

2. I. M. Longman, "The application of rational approximations to the solution of problems in theoretical seismology," *Bull. Seismological Soc. America*, v. 56, 1966, pp. 1045–1065.

3. P. F. Byrd & M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, New York, 1954. MR **15**, 702.

4. Y. L. Luke, "Simple formulas for the evaluation of some higher transcendental functions," *J. Math. Phys.*, v. 34, 1956, pp. 298–307. MR **17**, 1138.